Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3S9/M4S9

STOCHASTIC SIMULATION

Date: Tuesday, 30th May 2006 Time: 2 pm – 3.30 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

There are **four questions** only, and the exam lasts 1.5 hours.
Calculators may not be used.
Statistical tables will not be available.

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1. (a) A sequence of values U_i in $[0,1)$ is produced by the Fibonacci recursion

$$U_i = (U_{i-1} + U_{i-2}) \bmod 1.$$

Show that the sequence is not random, since the event $U_{i-2} < U_i < U_{i-1}$, $i \geq 3$, occurs with probability zero while in a truly random sequence this event occurs with probability $1/6$.

(b) Describe in detail and justify carefully the steps of the rejection sampling algorithm for generating random variables from a target density $f(x)$, using $U(0,1)$ random variables and random variables generated from a density $g(x)$.

A folded-normal variable X is defined by $X = |Z|$, where Z is distributed as standard normal, $Z \sim N(0,1)$.

Find the density function $f(x)$ of X.

It is proposed to simulate from $f(x)$ by rejection sampling using the envelope density $g(x) = \lambda \exp\{-\lambda x\}$. Explain how one might simulate from $g(x)$, and calculate the acceptance probability of the rejection sampling algorithm. Show that this acceptance probability is maximised by $\lambda = 1$.

2. (a) Write brief notes on the *antithetic variates* and *control variates* methods of variance reduction. What is meant by *importance sampling*?

(b) It is required to use Monte Carlo integration to estimate the value of

$$\theta = \int_0^1 \frac{e^x - 1}{e - 1} \mathrm{d}x.$$

An estimator $\widehat{\theta}_1$ of θ using $U_1, \ldots, U_n$ independently generated from $U(0,1)$ is constructed as

$$\widehat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{e^{U_i} - 1}{e - 1}.$$

Show that the variance of $\widehat{\theta}_1$ is $(2e - e^2/2 - 3/2)/\{n(e-1)^2\}$.

Suggest an estimator $\widehat{\theta}_2$ that might be constructed using antithetic variates U and $1-U$.

Suggest an estimator $\widehat{\theta}_3$ that might be constructed using U as a control variate.

Suggest and justify a density that might be used in constructing an importance sampling estimator of θ.

[You are **not** required to calculate the variances of your estimators $\widehat{\theta}_2$ and $\widehat{\theta}_3$].

3. (a) What is meant by a *Markov chain Monte Carlo* method? When are these methods particularly useful?

Describe in detail the *Gibbs sampler* algorithm for simulating from a high-dimensional posterior density $\pi(\theta \mid x)$, for a $d-$dimensional parameter θ.

(b) Describe in detail how the Gibbs sampler could be used to sample from a bivariate posterior density of the form

$$\pi(\theta_1, \theta_2 \mid x) \propto \exp\left\{-\frac{1}{2(1-\rho^2)}(\theta_1^2 - 2\rho\theta_1\theta_2 + \theta_2^2)\right\},$$

where ρ is known.

You should give detail of algorithms which might be used to perform the simulations required by the Gibbs sampler.

(c) Suppose we wish to use the Gibbs sampler to sample from the density

$$\pi(\theta_1, \theta_2 \mid x) \propto \exp\{-\frac{1}{2}(\theta_1 - 1)^2(\theta_2 - 2)^2\}.$$

Obtain the conditional distributions required by the Gibbs sampler algorithm.

Why does using the Gibbs sampler make no sense for this problem?

4. Write a detailed account, with careful definitions and, where appropriate, derivations, of **one** of the following:

(i) The ratio of uniforms method of generating random variates;

(ii) Bootstrap and jackknife methods of bias and variance estimation;

(iii) Monte Carlo tests;

(iv) The properties of congruential pseudo-random number generators.