1. (a) Let $X$ be a random variable with density $f(x) = \alpha\beta x^{\alpha-1}\exp(-\beta x^\alpha), x > 0$, with $\alpha > 0$, $\beta > 0$. Show how to generate $X$ from $U \sim U(0,1)$.

Let $X_1, \ldots, X_n$ be independent, identically distributed with density $f$ as above, and let $Z = \max\{X_1, \ldots, X_n\}$. Show how to generate $Z$ from a *single* $U(0,1)$ random variable $U$.

(b) Describe in detail the steps of the rejection sampling algorithm for generating random variables from a density $f(x)$, using $U(0,1)$ random variables and random variables generated from an envelope density $g(x)$. How is the algorithm modified if the density $f(x)$ is only specified up to a constant of proportionality, $f(x) \propto f^*(x)$?

Let $X$ have density $f(x) \propto x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$, where $0 < \alpha < 1, \beta > 1$. Devise an algorithm for generating $X$ by rejection sampling, using an envelope $g(x)$ proportional to $x^{\alpha-1}$. Find the acceptance probability for the algorithm. What happens to the acceptance probability as $\beta$ increases?

(c) Describe in detail what is meant by a *congruential pseudo-random number generator*. Write brief notes on the properties of such a generator.

2. (a) Let $h$ be a non-negative function with $\int h(x)dx < \infty$, and define

$$C_h = \left\{(u,v) :\ 0 \le u \le \sqrt{h(v/u)}\right\}.$$

Let $(U, V)$ be uniformly distributed on $C_h$. Derive the density function of $X = V/U$. Show that, under appropriate conditions on $h(x)$, the region $C_h$ may be bounded within a rectangle, which you should specify. Describe the steps of the 'ratio of uniforms' method for generating random variables with density proportional to $h(x)$.

Let $X$ be generated by the following algorithm:

(1) Generate $U_1, U_2 \sim U(0,1)$, set $V = 2U_2 - 1$.

(2) If $U_1^2 + V^2 > 1$, go to (1).

(3) Return $X = V/U_1$.

What is the distribution of $X$? Justify your answer carefully.

(b) Let $U$ be uniformly distributed on $(0,1)$. Show that for any monotonic function $g$ on $(0,1)$, $g(U)$ and $g(1-U)$ are negatively correlated.

Explain how this result is relevant to application of the *antithetic variates* method of variance reduction, which you should describe.

(c) Given a probability density function $f(x)$ and a function $\phi(x)$, explain what is meant by an *importance sampling estimator* of $\theta = \int \phi(x)f(x)dx$, which uses sampling from another density $g(x)$.

Show that the importance sampling estimator is unbiased, and find an expression for its variance. What choice for the density $g$ minimises this variance?

3. (a) Let $X_1, \ldots, X_n$ be independent, identically distributed from a population $F$ and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of the population parameter $\theta \equiv \theta(F)$.

Describe in detail both the *jackknife* and *bootstrap* estimators of the variance of $\hat{\theta}$, assuming a non-parametric problem.

Let $\theta = \mu$, where $\mu$ is the mean of $F$, and let the estimator $\hat{\theta}$ be $\hat{\theta} = \bar{X}$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Obtain the form of the jackknife and bootstrap variance estimators. Which estimator do you prefer, and why?

(b) Write brief notes on bootstrap procedures for constructing non-parametric confidence intervals for a population parameter $\theta$.

(c) Explain what is meant by a Monte Carlo test of size $\alpha$ of a null hypothesis $H_0$, where under $H_0$ the test statistic $T$ is a random variable with continuous distribution function $F$, not depending on any unknown parameters.

Let the Monte Carlo test be based on drawing $m$ random variates from $F$. Let $\beta^{(m)}(\alpha)$ be the power of the Monte Carlo test, and let $\beta(\alpha)$ be the power of the corresponding conventional significance test, under the alternative hypothesis $T \sim F_\theta$. Show that

$$\beta^{(m)}(\alpha) = \int_0^1 \beta(\xi) b(\xi) d\xi,$$

where $b(\xi)$ is the density of a particular beta distribution, which you should specify.

4. (a) What is meant by a *Markov chain Monte Carlo* method?

Describe in detail the *Gibbs sampler* and the *Metropolis-Hastings* algorithms, as used in Bayesian inference. Show that the Gibbs sampler may be viewed as a special case of the Metropolis-Hastings algorithm.

(b) Suppose that, given $\mu$ and $\tau$, $X$ is distributed as a $N(\mu, 1/\tau)$ random variable. Suppose further the prior assumptions that $\tau$ is distributed as a gamma random variable, with density proportional to $\tau^{\alpha-1} \exp(-\beta\tau)$, and that, given $\tau$, $\mu$ is distributed as $N(\nu, 1/\tau)$, where $\alpha, \beta, \nu$ are known constants.

Find the form of the posterior distribution of $(\mu, \tau)$, given the observation $x$ of $X$. Explain how to apply the Gibbs sampler to simulate from this posterior distribution.