UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M3S9 Stochastic Simulation

DATE: Monday, 9th June 2003 TIME: 10.00 am – 11.30 pm

There are four questions only, and the exam lasts $1\frac{1}{2}$ hours.

A project is set which carries the credit of the fifth question.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1. *a)* Describe in detail the Polar-Marsaglia algorithm for generating random variates, X, from a standard normal density, $X \sim N(0,1)$.

b) What makes the Polar-Marsaglia method more efficient than the Box-Muller approach?

c) Consider a rejection sampling approach to generating from the standard normal density,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty,$$

using a logistic density as the rejection envelope density,

$$g(x) = \frac{e^{-x/\beta}}{\beta(1+e^{-x/\beta})^2} \qquad -\infty < x < \infty, \quad \beta > 0.$$

i) For $\beta = 1$, show that the ratio $\frac{f(x)}{g(x)}$ is maximised at $x = 0$.

ii) For what values of β does the point $x = 0$ remain a maximum of the ratio $\frac{f(x)}{g(x)}$?

iii) Describe in detail the rejection algorithm for generating variates from $f(x)$ using $g(x)$ with $\beta = 1$.
You may assume that you have access to a stream of uniform random variates $U \sim U(0,1)$ and note that
$G(x) = \int_{-\infty}^{x} g(s)ds = (1 + \exp[-x])^{-1}$ *for* $\beta = 1$.

iv) Report the acceptance rate of the algorithm in *c) iii)*.
How does this compare with the acceptance rate of the Polar-Marsaglia method in *a)*?

2. *a)* Consider the region defined by

$$C_h = \left\{ (U,V) \;\middle|\; 0 \le U \le \sqrt{h\left(\frac{V}{U}\right)} \right\},$$

where $h(x) > 0 \;\forall\, x$, and $\int h(x)\,\mathrm{d}x < \infty$.

i) Show that C_h has finite area.

ii) If (U,V) are uniform on C_h, what is the probability density function of $X = \frac{V}{U}$?

b) It is required to generate random variates from the following Student distribution with three degrees of freedom,

$$h(x) \;\propto\; \frac{1}{\left(1+\frac{1}{3}x^2\right)^2} \qquad -\infty < x < \infty.$$

By bounding the region defined by C_h within a rectangle, whose size you should specify, describe the steps of the ratio of uniforms algorithm to generate variates from $h(x)$.

c) Give two general methods that can be used for improving the efficiency of the ratio of uniforms method.

3. Consider the following integral:

$$\theta = \int_0^1 \frac{e^u - 1}{e - 1}\, \mathrm{d}u \;=\; \int_0^1 \phi(u) f(u)\, \mathrm{d}u.$$

a) Write down an expression for the Monte Carlo estimator, $\widehat{\theta}_1$, of θ, using uniform random variates, $U_1, \ldots, U_n \sim U(0,1)$. That is, $f(u)$ is the uniform density on $[0,1]$. Determine an expression for the variance of the estimator $\widehat{\theta}_1$.

b) Give an expression for an estimator, $\widehat{\theta}_2$, of θ, composed using antithetic variates U_i and $1 - U_i$, $i = 1, \ldots, n$.
State why antithetic variates are suitable for estimating θ and explain why $\widehat{\theta}_2$ will have lower variance than $\widehat{\theta}_1$.

c) Determine the variance of the estimator, $\widehat{\theta}_3$, of θ, composed using U as the control variate, such that,

$$\widehat{\theta}_3 = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{e^{U_i} - 1}{e - 1} - \beta[U_i - \mathrm{E}(U_i)] \right\}.$$

What value of β minimizes this variance?

4. *a)* *i)* Outline the Metropolis algorithm for obtaining samples from a discrete distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ using a symmetric proposal distribution with transition matrix Q.

ii) Let P denote the $k \times k$ one step transition matrix for the resulting Markov chain in *a) i)*. Provide expressions for the elements of P in terms of π and Q.

iii) Show that $\pi = \pi P$.

iv) What are the implications of the result in *a) iii)*?

b) *i)* Outline the Gibbs sampler for obtaining samples from a bivariate discrete distribution, $\pi(X, Y)$.

ii) Suppose X and Y are binary, $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ with the joint distribution $\pi(X, Y)$,

$$\begin{bmatrix} \pi(0,0) & \pi(1,0) \\ \pi(0,1) & \pi(1,1) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.3 \end{bmatrix}.$$

Consider an initial sample drawn from this distribution, $\{X^{(0)}, Y^{(0)}\} \sim \pi(X, Y)$. Show that following one iteration of the Gibbs sampler, $\{X^{(0)}, Y^{(0)}\} \to \{X^{(1)}, Y^{(1)}\}$, the distribution of the new state $\{X^{(1)}, Y^{(1)}\}$ is unchanged, $\{X^{(1)}, Y^{(1)}\} \sim \pi(X, Y)$.