Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) Determine whether the following process is stationary, giving your reasons.

$$X_t + \frac{1}{12}X_{t-1} = \frac{1}{24}X_{t-2} + \epsilon_t.$$

   (c) Define a real-valued deterministic sequence $\{y_t\}$ by

$$y_t = \begin{cases} +1, & \text{if } t = 0,\ -1,\ -2,\ \ldots, \\ -1, & \text{if } t = 1,\ 2,\ 3,\ \ldots. \end{cases}$$

   Now define a stochastic process by $X_t = y_t I$, where I is a random variable taking on the values $+1$ and -1 with probability $1/2$ each.

   Find the mean, variance and autocovariance of $\{X_t\}$ and determine, with justification, whether this process is stationary.

   (d) A complex-valued time series Z_t is given by $Z_t = C\mathrm{e}^{\mathrm{i}(2\pi f_0 t+\theta)}$, where f_0 and C are finite real-valued constants and θ is uniformly distributed over $[-\pi, \pi]$.

   Determine, with justification, whether this process is stationary.

   [*The autocovariance for a complex-valued time series is given by* $\mathrm{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t Z_{t+\tau}^*\} - E\{Z_t\}E\{Z_{t+\tau}^*\}$, *where* * *denotes complex conjugate.*]

2. (a) Suppose $\{X_t\}$ is an MA(q) process with zero mean, i.e., X_t can be expressed in the form

$$X_t = -\theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$). Show that its autocovariance sequence is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \leq q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

(b) Consider the non-invertible MA(2) process

$$X_t = \epsilon_t - \frac{9}{4}\epsilon_{t-2}$$

with $\theta_{1,2} = 0$.

(i) Calculate the *autocorrelation* sequence of this process.

(ii) Find an *invertible* MA(2) process having the same autocorrelation sequence, fully justifying your result.

(c) Suppose that $\{X_t\}$ is the MA(2) process

$$X_t = \epsilon_t - \theta_{2,2}\epsilon_{t-2}$$

with $\theta_{1,2} = 0$.

Now let $Y_t = X_{2t}, t \in \mathbb{Z}$, i.e., the process $\{Y_t\}$ is formed by subsampling every other random variable from the process $\{X_t\}$, and hence $\{Y_t\}$ has a sampling interval of 2.

Given that $s_{Y,\tau} = s_{X,2\tau}$ show that $S_Y(f) = 2S_X(f)$ for $|f| \leq 1/4$.

Hint: A stationary process with autocovariance sequence $\{s_\tau\}$ and sample interval Δt has spectral density function

$$S(f) = \Delta t \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau\Delta t}.$$

3. (a) Use the fact that

$$(1-z)\sum_{t=-(N-1)}^{N-1} z^t = z^{-N+1} - z^N$$

to show that

$$\sum_{t=-(N-1)}^{N-1} \mathrm{e}^{\mathrm{i}2\pi f t} = (2N-1)\mathcal{D}_{2N-1}(f),$$

where $\mathcal{D}_{2N-1}(f)$ is a form of Dirichlet's kernel, defined as

$$\mathcal{D}_{2N-1}(f) = \frac{\sin[(2N-1)\pi f]}{(2N-1)\sin(\pi f)}.$$

(b) Consider the following autocovariance sequence,

$$s_\tau = \begin{cases} 1, & \text{if } |\tau| \le K, \\ 0, & \text{if } |\tau| > K, \end{cases}$$

where $K \ge 2$. Is $\{s_\tau\}$ the autocovariance sequence for some discrete stationary process with spectral density function $S(f)$?

(c) Specify the three conditions which must be satisfied by a linear time-invariant (LTI) digital filter.

(d) Let $\{X_t\}$ be a discrete stationary process with a spectral density function $S_X(f)$. Let

$$Y_t = X_t - \frac{1}{2K+1}\sum_{j=-K}^{K} X_{t+j},$$

where $K \ge 1$.

Find the spectral density function, $S_Y(f)$, for $\{Y_t\}$ when $\{X_t\}$ is white noise with variance unity.

4. (a) What is meant by saying two discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly stationary stochastic processes?

(b) Suppose $\{X_t\}$ and $\{Y_t\}$ are zero mean jointly stationary processes given by

$$X_t = \epsilon_t + \theta\epsilon_{t-1}; \qquad Y_t = \epsilon_t - \theta\epsilon_{t-1},$$

with $|\theta| < 1$.

(i) By first finding the cross-covariance sequence $\{s_{XY,\tau}\}$, or otherwise, show that the cross-spectrum $S_{XY}(f)$ is given by

$$S_{XY}(f) = \sigma_\epsilon^2[(1-\theta^2) + 2\mathrm{i}\theta\sin 2\pi f].$$

(ii) Find the value of the magnitude squared coherence, $\gamma_{XY}^2(f)$.

(iii) Now consider $\{X_t\}$ to be the input to a linear filter with frequency response function $G(f)$ and $\{Y_t\}$ to be the output.
Find $|G(f)|^2$ and hence identify the polynomials $\Phi(B)$ and $\Theta(B)$ in the stationary and invertible ARMA representation

$$\Phi(B)Y_t = \Theta(B)X_t.$$

Here, as usual, B is the backshift operator.

(iv) Explain the magnitude squared coherence value obtained in (ii) in terms of the result in (iii).

5. Assume that $\{X_t\}$ can be written as a one-sided linear process, so that

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t.$$

We wish to construct the l-step ahead forecast

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k} = \delta(B)\varepsilon_t.$$

(a) Show that the l-step prediction variance $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

(b) Consider the stationary AR(2) process $X_t = \phi_{2,2}X_{t-2} + \epsilon_t$, where $\phi_{1,2} = 0$. Show that

$$X_t(l) = \begin{cases} \phi_{2,2}^{l/2} X_t, & \text{if } l \text{ even,} \\ \phi_{2,2}^{(l+1)/2} X_{t-1}, & \text{if } l \text{ odd.} \end{cases}$$

(c) It was stated in the course notes that for a general AR(p) process,

$$X_t = \phi_{1,p}X_{t-1} + \ldots + \phi_{p,p}X_{t-p} + \epsilon_t,$$

that $X_t(l)$ depends only on the last p observed values of $\{X_t\}$ and may be obtained by solving the AR(p) difference equation with the future innovations set to zero; in particular

$$X_t(1) = \phi_{1,p}X_t + \ldots + \phi_{p,p}X_{t-p+1}$$

which is X_{t+1} with the future innovation set to zero.

(i) Show that, for $p \geq 2$,

$$X_{t+2} = \phi_{1,p}\left[X_t(1) + \epsilon_{t+1}\right] + \sum_{j=2}^{p} \phi_{j,p}X_{t+2-j} + \epsilon_{t+2}$$

and hence that

$$X_{t+2} = \sum_{j=1}^{p} [\phi_{1,p}\phi_{j,p} + \phi_{j+1,p}]X_{t+1-j} + \phi_{1,p}\epsilon_{t+1} + \epsilon_{t+2},$$

where $\phi_{j,p} = 0$ if $j > p$.

(ii) What does the result in (i) suggest for $X_t(2)$? Show that this $X_t(2)$ agrees with the result in (b).