Imperial College London

**Department of Mathematics**

**BSc and MSci EXAMINATIONS (MATHEMATICS) MAY–JUNE 2006**

*This paper is also taken for the relevant examination for the Associateship.*

**M3S8/M4S8 Time Series**

DATE: Wednesday, 24th May 2006 TIME: 10 am – 12

*Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.*

*Calculators may not be used. Statistical tables will not be available.*

**Note:** Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance $\sigma_\epsilon^2$, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise.

**1.** *a)* *i)* What is meant by saying that a stochastic process is stationary?

*ii)* For a stationary stochastic process $\{Y_t\}$ with non-zero mean, variance $\sigma^2$, and autocovariance sequence $s_\tau$, show that

$$E\{(Y_{t+\tau} - Y_t)^2\} = 2[\sigma^2 - s_\tau].$$

*b)* Consider the following MA(2) process:

$$X_t = \epsilon_t - \frac{5}{6}\epsilon_{t-1} + \frac{1}{6}\epsilon_{t-2}.$$

*i)* Is this MA(2) process invertible?

*ii)* What does it mean to say that a moving average process is invertible?

*iii)* Now let $Y_t = X_t + a_t$, where $\{a_t\}$ is a white noise process with zero mean and variance $\sigma_a^2$, uncorrelated with $\{\epsilon_t\}$. Show that $\rho_{Y,\tau} < \rho_{X,\tau}$, for all $\tau$, where $\{\rho_{Y,\tau}\}$ and $\{\rho_{X,\tau}\}$ are the autocorrelation sequences of the processes $\{Y_t\}$ and $\{X_t\}$, respectively.

*c)* Consider the AR(2) process:

$$X_t - \phi_{1,2}X_{t-1} - \phi_{2,2}X_{t-2} = \epsilon_t.$$

where $\phi_{1,2} = -\alpha, \phi_{2,2} = -1$ and $\alpha \neq 2$.

*i)* What is the product of the roots of the corresponding characteristic equation, and what does this imply about the roots relative to the unit circle?

*ii)* Is this AR(2) process stationary?

**2.** Let

$$X_t = \cos(2\pi f_0 t + \phi) + \epsilon_t,$$

where $f_0$ is a fixed frequency.

*a)* Show that $\{X_t\}$ is non-stationary if $\phi$ is constant.

*b)* Now suppose henceforth that $\phi$ is uniformly distributed on $(-\pi, \pi)$ and is independent of $\{\epsilon_t\}$.

*i)* Find the mean, variance and covariance $s_\tau$ of $\{X_t\}$ and hence prove that $\{X_t\}$ is stationary.

*ii)* Contrast the behaviour of the autocovariance sequence for $\{X_t\}$ with the behaviour of the autocovariance sequence of a process with a purely continuous spectrum.

*iii)* By looking at the form of $\{X_t\}$, describe the form of the spectrum of $\{X_t\}$ and classify it as one of: purely continuous, purely discrete, discrete or mixed.

**3.** *a)* A certain type of series $\{Y_t\}$ can be modelled as the sum of a primary signal $\{X_t\}$ plus a delayed echo of that signal:

$$Y_t = X_t + aX_{t-d},$$

where $d > 0$ is an integer constant, and $a$ is a real constant.

*i)* Show that the filter defined by $L\{X_t\} = X_t + aX_{t-d}$ satisfies the three defining properties of a linear time-invariant digital filter.

*ii)* Show that

$$S_Y(f) = \left[1 + a^2 + 2a\cos(2\pi f d)\right] S_X(f),$$

where $S_Y(f)$ and $S_X(f)$ are the spectral density functions of $\{Y_t\}$ and $\{X_t\}$, respectively.

*b)* Consider the ARMA$(p, q)$ process

$$\Phi(B)X_t = \Theta(B)\epsilon_t.$$

*i)* Derive the form of its spectral density function $S(f)$.

*ii)* Hence, or otherwise, identify the *stationary* and *invertible process* $\{X_t\}$ having the spectral density

$$S(f) = \frac{5 - 4\cos 2\pi f}{10 + 6\cos 2\pi f},$$

obtaining values for any autoregressive parameters $\{\phi_{j,p}\}$, any moving average parameters $\{\theta_{j,q}\}$, and $\sigma_\epsilon^2$.
*[Hint: Think carefully about the roots of any autoregressive and moving average characteristic polynomials.]*

**4.** *a)* *i)* What is meant by saying two discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are jointly stationary stochastic processes?

*ii)* Suppose $\{X_t\}$ and $\{Y_t\}$ are zero mean stationary processes such that $\{Y_t\}$ is a linear filtering of $\{X_t\}$ plus zero mean stationary noise:

$$Y_t = \sum_{l=-\infty}^{\infty} a_l X_{t-l} + \eta_t,$$

where $\{a_l\}$ is a real-valued filter sequence, and $\{\eta_t\}$ is noise, uncorrelated with $\{X_t\}$. Use the spectral representations of $\{X_t\}, \{Y_t\}$ and $\{\eta_t\}$ to show that the spectra are related by

$$S_Y(f) = |A(f)|^2 S_X(f) + S_\eta(f),$$

where $A(f)$ is the transfer function or frequency response function of $\{a_l\}$, and $S_Y(f), S_X(f)$ and $S_\eta(f)$ are the spectral density functions, (all assumed to exist), of $\{Y_t\}, \{X_t\}$ and $\{\eta_t\}$, respectively.

*b)* Two zero mean stationary processes are defined in terms of a filtering of the zero mean stationary process $\{X_t\}$ plus zero mean stationary noise:

$$Y_t = \sum_{l=-\infty}^{\infty} a_l X_{t-l} + \eta_t \qquad \text{and} \qquad Z_t = \sum_{m=-\infty}^{\infty} b_m X_{t-m} + \nu_t,$$

where $\{b_m\}$ is a real-valued filter sequence, the noise $\{\eta_t\}$ is uncorrelated with $\{X_t\}$, and the noise $\{\nu_t\}$ is uncorrelated with both $\{X_t\}$ and $\{\eta_t\}$, and has spectrum $S_\nu(f)$.

*i)* Show that the cross-covariance sequence $\{s_{YZ,\tau}\}$ is given by

$$s_{YZ,\tau} = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_l b_m s_{X,\tau-m+l},$$

where $\{s_{X,\tau}\}$ is the autocovariance sequence of $\{X_t\}$.

*ii)* Hence show that the cross-spectrum $S_{YZ}(f)$ is given by

$$S_{YZ}(f) = A^*(f) B(f) S_X(f),$$

where $B(f)$ is the transfer function or frequency response function of $\{b_m\}$.

*iii)* Find the value of the magnitude squared coherence, $\gamma_{YZ}^2(f)$, when $S_\eta(f)$ and $S_\nu(f)$ are zero, and $S_X(f), S_Y(f)$ and $S_Z(f)$ are non-zero, for all $|f| \leq 1/2$. Explain why $\gamma_{YZ}^2(f)$ takes this value in this case.

**5.** *a)* *i)* Suppose $\{X_t\}$ has an AR($p$) representation:

$$X_t - \phi_{1,p}X_{t-1} - \ldots - \phi_{p,p}X_{t-p} = \epsilon_t.$$

Show that if $s_j$ denotes the lag-$j$ autocovariance, then

$$\boldsymbol{\gamma}_p = \Gamma_p \boldsymbol{\phi}_p \quad \text{and} \quad \sigma_\epsilon^2 = s_0 - \sum_{j=1}^{p} \phi_{j,p} s_j,$$

where $\boldsymbol{\gamma}_p = [s_1, s_2, \ldots, s_p]^T$; $\boldsymbol{\phi}_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}.$$

*ii)* Find the parameter values $\phi_{1,2}, \phi_{2,2}$ and $\sigma_\epsilon^2$ of the stationary AR(2) process for which $s_0 = 21/10, s_1 = 3/2$ and $s_2 = 9/10$.

*b)* Suppose that a realization $X_1, \ldots, X_N$ is available of a stationary Gaussian AR($p$) process. The vector $\widehat{\boldsymbol{\phi}}_{FB}$ that minimizes

$$SS_F(\boldsymbol{\phi}) + SS_B(\boldsymbol{\phi}),$$

where $SS_F(\boldsymbol{\phi})$ is the forward sum of squares, and $SS_B(\boldsymbol{\phi})$ is the backward sum of squares, is called the forward/backward least squares estimator for the $\{\phi_{j,p}\}$.

*i)* Why does it make sense to use the combined forward/backward method, rather than just forward or backward least squares for parameter estimation?

*ii)* For an AR(1) process, show that the forward/backward least squares estimator of $\phi_{1,1}$ is given by

$$\widehat{\phi}_{1,1;FB} = \frac{\sum_{t=2}^{N} X_t X_{t-1}}{\frac{1}{2}(X_1^2 + X_N^2) + \sum_{t=2}^{N-1} X_t^2},$$

*iii)* Find the forward least squares estimator $\widehat{\phi}_{1,1;F}$ and backward least squares estimator $\widehat{\phi}_{1,1;B}$, and compare with the forward/backward least squares estimator. Interpret the differences.

*iv)* For the unrealistic case of $N = 2$, show that if $X_1 \neq X_2$, then $|\widehat{\phi}_{1,1;FB}| < 1$, while $\widehat{\phi}_{1,1;F}$ and $\widehat{\phi}_{1,1;B}$ can be arbitrarily large in magnitude.