1. Assume that $\{X_t\}_{t=-\infty}^{\infty}$ follows the ARCH(1) model:

$$\begin{aligned} X_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= a_0 + a_1 X_{t-1}^2, \end{aligned}$$

where a_0 is a positive constant, a_1 is a constant $\in (0, 3^{-1/2})$, and $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of independent, Normally distributed random variables with mean zero and variance one. (Recall that if $Z \sim N(0, \sigma^2)$, then $\mathbb{E}(Z^4) = 3\sigma^4$.) The above conditions guarantee that X_t is completely stationary with $\mathbb{E}(X_t^4) < \infty$.

(a) Compute $\mathbb{E}(X_t^2)$ as a function of a_0 and a_1.

(b) Compute $\mathbb{E}(X_t^4)$ as a function of a_0 and a_1.

(c) For a zero-mean random variable Y with $\mathbb{E}(Y^4) < \infty$, its *kurtosis* κ_Y is defined as

$$\kappa_Y = \frac{\mathbb{E}(Y^4)}{(\mathbb{E}(Y^2))^2}.$$

Compute the kurtosis of X_t as a function of a_1. Use the result to deduce that X_t is not Normally distributed.

(d) Compute the autocorrelation coefficient of X_t^2 at lag 1.

2. Assume that $\{X_t\}_{t=-\infty}^{\infty}$ is a stationary stochastic process which admits an AR(2) representation

$$X_t = aX_{t-1} + bX_{t-2} + \varepsilon_t,$$

where a and b are real-valued constants and $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of real-valued uncorrelated random variables with mean zero and variance one. Suppose that we observe $\mathbf{x} = (x_1, x_2, \ldots, x_n)$: a finite realisation of the above process X_t.

(a) Given $\mathbf{x}$, compute $\hat{a}^{YW}$ and $\hat{b}^{YW}$: the Yule-Walker estimates of a and b, respectively.

(b) Given $\mathbf{x}$, compute $\hat{a}^{LS}$ and $\hat{b}^{LS}$: the least squares estimates of a and b, respectively.

Note: represent $\hat{a}^{YW}$, $\hat{b}^{YW}$, $\hat{a}^{LS}$ and $\hat{b}^{LS}$ as functions of $(x_1, \ldots, x_n)$ only. Your final answer should *not* contain any terms of the form A^{-1} where A is a matrix (of size other that 1×1), or AB where A and B are matrices and/or vectors (of sizes other than 1×1).

(c) Which estimation method (Yule-Walker or least squares) would you use in the cases

(i) $n = 3$?

(ii) $n = 10^{10}$?

Why?

3. Let $\{a_t\}_{t=-\infty}^{\infty}$ be a sequence of independent random variables, distributed Uniformly on $[0,1]$, and let $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ be a sequence of independent Normal random variables with mean zero and variance one. Further, let the processes $\{a_t\}_{t=-\infty}^{\infty}$ and $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ be independent of each other. Define $\{X_t\}_{t=-\infty}^{\infty}$ as a "random coefficient" AR(1) process:

$$X_t = a_t X_{t-1} + \sqrt{\frac{8}{9}}\varepsilon_t.$$

You may assume (without proof) that $\{X_t\}_{t=-\infty}^{\infty}$ is second-order stationary.

(a) Compute the autocovariance sequence of X_t.

(b) Prove or disprove the following statement: the autocovariance sequence of X_t is the same as that of the AR(1) process Y_t, defined by

$$Y_t = \frac{1}{2}Y_{t-1} + \varepsilon_t.$$

(c) Compute the spectral density of X_t (your final answer should *not* contain complex exponentials).

(d) Prove or disprove the following statement: for all integers $n \geq 1$, $t_1, t_2, \ldots, t_n$, and τ, the distribution of the random vector $\{X_{t_i}\}_{i=1}^{n}$ is the same as the distribution of the random vector $\{Y_{t_i+\tau}\}_{i=1}^{n}$. Hint: you may want to consider moments of X_t and Y_t of order higher than 2.

4. Let $\{X_t\}_{t=-\infty}^{\infty}$ be a second-order stationary stochastic process with mean μ and autocovariance sequence s_τ. Suppose that we observe $\mathbf{x} = (x_1, x_2, \ldots, x_n)$: a finite realisation of the above process X_t.

(a) The sample mean estimator of μ is defined as

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} x_t.$$

Show that if s_τ is given by $s_0 = \pi^2/6$ and $s_\tau = \frac{1}{|\tau|}(1 + \frac{1}{2} + \ldots + \frac{1}{|\tau|})$ for $\tau \neq 0$, then the mean-square error of $\hat{\mu}$ tends to zero as n tends to infinity.

(b) For $|\tau| < n$, the sample autocovariance estimator of s_τ is defined as

$$\hat{s}_\tau = \frac{1}{n}\sum_{t=1}^{n-|\tau|}(x_t - \hat{\mu})(x_{t+|\tau|} - \hat{\mu}).$$

Show that

$$\sum_{|\tau|<n} \hat{s}_\tau = 0.$$

(c) Assume now that $\mu = 0$ and that X_t has a spectral density $S(f)$. We define the periodogram at frequency $f \in [-1/2, 1/2]$ as

$$\hat{S}(f) = \frac{1}{n}\left|\sum_{t=1}^{n} x_t e^{-i2\pi f t}\right|^2.$$

Using the spectral representation theorem, represent $\mathbb{E}(\hat{S}(f))$ in the form

$$\mathbb{E}(\hat{S}(f)) = \int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')df'$$

for an appropriate function $\mathcal{F}$. You may leave your answer in terms of complex exponentials.

(d) Show that the sequence $\{s_\tau\}_{\tau=-\infty}^{\infty}$, defined in Part (a) above, is in fact a valid autocovariance sequence (i.e. that there exists a stochastic process whose autocovariance sequence is s_τ). Hint: $\pi^2/6 = \sum_{k=1}^{\infty} k^{-2}$.

5. Assume that $\{X_t\}$ can be written as a one-sided linear process, so that

$$X_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} = \Psi(B)\varepsilon_t,$$

where $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ is a white noise sequence with mean zero and variance one. We wish to construct the l-step ahead forecast

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k} = \delta(B)\varepsilon_t.$$

(a) Show that the l-step prediction variance $\sigma^2(l) = \mathbb{E}\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

(b) Consider the following stationary AR(1) model,

$$X_t = aX_{t-1} + \varepsilon_t,$$

where a is a real-valued constant. For $l = 1$, show that setting $\delta_k = \psi_{k+l}$ is equivalent to setting

$$X_t(1) = aX_t.$$

(c) In the above, $X_t(1) = aX_t = \mathbf{a}^T\mathbf{X}$, where

$$\begin{aligned} \mathbf{a}^T &= (a, 0, \ldots, 0) \\ \mathbf{X}^T &= (X_t, X_{t-1}, \ldots, X_{t-n+1}). \end{aligned}$$

Denoting $s_\tau = \text{cov}(X_t, X_{t+\tau})$, show that

$$\mathbf{a} = \Gamma_{(n)}^{-1}\gamma_{(n)},$$

where $\Gamma_{(n)}$ is the $(n \times n)$ variance-covariance matrix of X_t, and

$$\gamma_{(n)}^T = (s_1, \ldots, s_n).$$

You do not need to prove the existence of $\Gamma_{(n)}^{-1}$ (i.e. to show that $\Gamma_{(n)}$ is invertible).