Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8 Time Series

Date: Thursday, 10th June 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables having zero mean and variance σ_ϵ^2, unless stated otherwise.

1. **(a)** What is meant by saying that a stochastic process is second-order stationary?

(b) Determine, with reasoning, which of the following processes are second-order stationary

(i) $X_t = \epsilon_t - 0.9\epsilon_{t-1}$.

(ii) $X_t = \frac{9}{4}X_{t-1} - \frac{9}{8}X_{t-2} + \epsilon_t$.

(c) Consider the following process,

$$X_t = \alpha X_{t-1} + \epsilon_t + \alpha\epsilon_{t-1},$$

with $|\alpha| < 1$.

(i) Show that

$$\text{var}\{X_t\} = \frac{\sigma_\epsilon^2(1+3\alpha^2)}{1-\alpha^2}.$$

(ii) Determine, ρ_1, the first autocorrelation of $\{X_t\}$.

2. Consider the following two alternative models for a particular time series

$$\begin{array}{ll} \mathbf{A}: & X_t = 0.89X_{t-1} + 0.1X_{t-2} + \epsilon_t. \\ \mathbf{B}: & X_t = X_{t-1} + \epsilon_t - 0.1\epsilon_{t-1}. \end{array}$$

(a) The infinite Auto-Regressive representation for a time series is given by,

$$X_t = \sum_{k=1}^{\infty} \pi_k X_{t-k} + \epsilon_t$$

(i) Determine π_1, π_2 and π_3 for the two models $\mathbf{A}$ and $\mathbf{B}$.

(ii) Give the general form of π_k for model $\mathbf{B}$.

(b) **(i)** What are the three properties of a linear time-invariant (LTI) filter?

(ii) Using LTI filters associated with their Auto-Regressive representations, determine the spectral density functions of the two models $\mathbf{A}$ and $\mathbf{B}$ (leaving your answers in terms of complex exponentials).

(c) Given your answers to **(a)** and **(b)** comment on the "difference" between the two models.

3. Given a sample $X_1, \ldots, X_N$ from a zero mean process with autocovariance function s_τ and spectral density funtion $S(f)$, we define $\widehat{s}_\tau^{(p)}$ – the biased estimator of s_τ and $\widehat{S}^{(p)}(f)$ – the associated estimator of $S(f)$ as

$$\widehat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|} \qquad \widehat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N} X_t e^{-i2\pi f t}\right|^2,$$

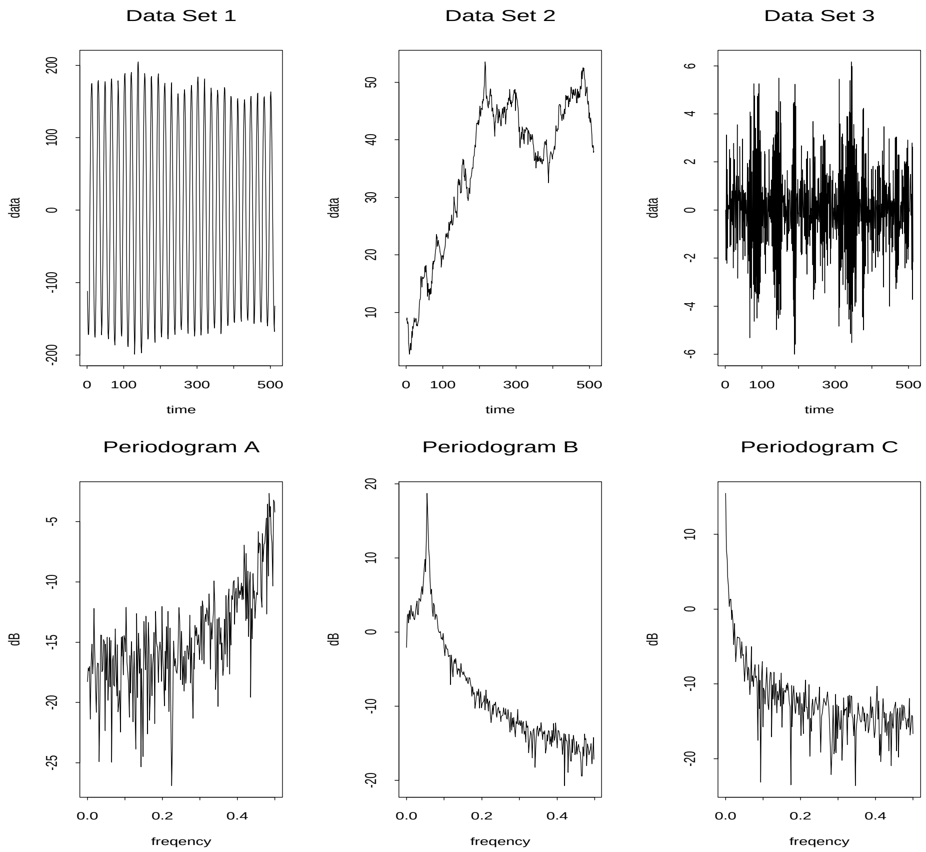
for $|\tau| = 0, 1, \ldots, N-1$ and $|f| \le 1/2$, respectively. $\widehat{S}^{(p)}(f)$ is known as the periodogram.

(a) Show that $\widehat{s}^{(p)}$ is a biased estimator of s_τ.
Why might we prefer this estimator to one which is unbiased?

(b) Show that

$$\mathsf{E}\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')\,df' \quad \text{where} \quad \mathcal{F}(f) = \left|\sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi f t}\right|^2.$$

(c) Consider the following three datasets and three periodograms. With justification, identify which periodogram is associated with which dataset.



(d) Under what circumstances may the periodogram be a biased estimator of the spectral density function? Which of the periodograms in part **(c)** may be biased? Describe in detail how tapering may reduce this potential bias (you should include an evaluation of the expected value of the tapered estimator).

4. Consider the following process

$$X_t = \phi X_{t-6} + \epsilon_t.$$

where $\phi \neq 0$ and $|\phi| < 1$.

(a) Show that the autocovariance sequence of $\{X_t\}$ is only non-zero for lags that are multiples of six.

(b) Using a sample $X_1, \ldots, X_N$ from this process.

(i) Determine the Yule-Walker estimators $\widehat{\phi}$ of ϕ and $\widehat{\sigma_\epsilon^2}$ of σ_ϵ^2 based on $\widehat{s}_0^{(p)}$ and $\widehat{s}_6^{(p)}$ (as defined in Question **3**).

(ii) The forward least squares model can be formulated as

$$X_F = F\phi + \epsilon_F$$

$$X_F = (X_7, \ldots, X_N)^\top; \;\; F = (X_1, \ldots, X_{N-6})^\top; \;\; \epsilon_F = (\epsilon_7, \ldots, \epsilon_N)^\top.$$

Determine the forward least squares estimators $\widetilde{\phi}$ of ϕ and $\widetilde{\sigma_\epsilon^2}$ of σ_ϵ^2.

(iii) Describe the relationship between the Yule-Walker and forward least squares estimators as N increases.

5. Assume that $\{X_t\}$ can be written as a one-sided linear process, so that

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t.$$

We wish to construct the l-step ahead forecast

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \delta(B)\epsilon_t.$$

(a) Show that the l-step prediction variance $\sigma^2(l) = \mathsf{E}\{(X_{t+l} - X_t(l))^2\}$ is minimized by setting $\delta_k = \psi_{k+l}$.

(b) Consider the following AR(1) model,

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t.$$

(i) For $l = 1$ and $l = 2$, show that setting $\delta_k = \psi_{k+l}$ is equivalent to setting

$$X_t(1) = \frac{1}{2}X_t \quad \text{and} \quad X_t(2) = \frac{1}{4}X_t.$$

(ii) Find the 1- and 2-step prediction variance: $\sigma^2(1)$ and $\sigma^2(2)$, when $\delta_k = \psi_{k+l}$.

(iii) Describe, specifically for this model, how we can produce a new forecast $X_{t+1}(1)$ of X_{t+2}, by updating $X_t(2)$ using $X_t(1)$ and X_{t+1} once it has been observed. [*Hint: start by showing that* $X_{t+1}(1) = X_t(2) + \psi_1 \epsilon_{t+1}$]