UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8 TIME SERIES

DATE: Tuesday, 27th May 2003 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables having zero mean and variance σ_ϵ^2, unless stated otherwise.

1. *a)* What is meant by saying that a stochastic process is second-order stationary?

b) Write down a condition, involving the roots of an equation, for an AR(p) (Autoregressive, order p) process to be second order stationary.

c) Consider the following AR(2) model,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t.$$

i) Using the backward shift operator B, show that this model can be written as the following infinite MA (Moving Average) process,

$$X_t = \sum_{k=0}^{\infty} \frac{\rho_1^{k+1} - \rho_2^{k+1}}{\rho_1 - \rho_2} \epsilon_{t-k},$$

where,

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \rho_1 B)(1 - \rho_2 B).$$

ii) Use the expression derived in part *c) i)* to verify that the condition given in part *b)* does indeed imply second order stationarity for $\{X_t\}$.

2. The process $\{X_t\}$ is given by,

$$X_t = \alpha + \beta t + Y_t,$$

where $\{Y_t\}$ is a second-order stationary process.

a) Show that differencing twice can be used to produce a new process $Z_t = \Delta^2 X_t = (1-B)^2 X_t$, which does not depend on α or β.

Determine whether or not $\{Z_t\}$ is second-order stationary.

b) *i)* What are the three properties of a linear time-invariant (LTI) filter?

ii) Determine the transfer function associated with the differencing operation carried out in *a)*, and use this to determine the spectrum of $S_Z(f)$ in terms of $S_Y(f)$.

iii) Given that $\{Y_t\}$ is an MA(2) process given by,

$$Y_t = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}.$$

Determine $S_Z(f)$ in terms of $\theta_{1,1}, \theta_{2,2}$ and σ_ϵ^2 (you may leave your answer in terms of complex exponentials).

3. *a)* What is meant by saying that a stochastic process is invertible?

b) Show that any stationary AR(p) process is invertible.

c) Write down a condition, involving the roots of an equation, for an MA(q) process to be invertible.

d) *i)* Determine the autocovariance sequence $s_\tau, \tau = 0, \pm1, \pm2, \ldots$ for the MA(1) process,

$$X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1}.$$

ii) When modelling a process $\{X_t\}$ as an MA(1) process, show that there is more than one such process with $s_0 = 5.0$ and $s_1 = 2.0$. Verify that only one of these processes is invertible.

4. *a)* Determine the spectral density function for the process $\{Z_t\}$ defined by,

$$Z_t = 0.99 Z_{t-3} + \epsilon_t.$$

Does the spectral density suggest that $\{Z_t\}$ will exhibit periodic behaviour, and if so, what is the approximate frequency of the oscillation?

b) Show that the zero-mean stationary process $\{X_t\}$ with spectral density function,

$$S(f) = 4\sigma^2\left(\frac{1}{2} - |f|\right) \qquad |f| \le \frac{1}{2},$$

has autocovariance sequence,

$$s_\tau = \begin{cases} \sigma^2 & \tau = 0; \\ \dfrac{4\sigma^2}{(\pi\tau)^2} & |\tau| = 1, 3, 5, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

5. *a)* Given a sample, $X_1, \ldots, X_N$ of a zero-mean discrete stationary time series $\{X_t\}$, the periodogram, $\widehat{S}^{(p)}(f)$, an estimator of the spectral density function $S(f)$, is given by,

$$\widehat{S}^{(p)}(f) = |J(f)|^2 = J(f)J^*(f); \quad J(f) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t e^{-i2\pi f t}.$$

i) Using the spectral representation theorem for X_t, show that the expected value of the periodogram can be written,

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')\,\mathrm{d}f'.$$

where $\mathcal{F}(f) = \left|\sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi f t}\right|^2$.

ii) Under what circumstances is the periodogram likely to produce a biased estimator of $S(f)$?

What properties of $\mathcal{F}(f)$ relate to this bias?

b) *i)* Describe how a real-valued taper $\{h_1, \ldots, h_N\}$ can be used to provide an alternative estimator, $\widehat{S}^{(d)}(f)$, of $S(f)$.

Under what circumstances may such an estimator have better bias properties than the periodogram?

ii) If $\{X_t\}$ is a white noise process with variance σ_X^2, determine a constraint on the values of the taper such that $\widehat{S}^{(d)}(f)$ is an unbiased estimator of $S(f)$ at $f = 0$.