Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M3S4/M4S4

Applied Probability

Date: Tuesday, 29th May 2007 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are included on pages 5 & 6.

1. (i) Write down the three axioms of a Poisson process with rate λ .

(ii) Let $X(t_1, t_2)$ be the number of events which occur in the interval $[t_1, t_2)$ for a Poisson process with rate λ. Let $p_0(t) = \mathsf{P}(X(0,t) = 0)$. By expressing $p_0(t + \delta t)$ in terms of $\mathsf{P}(X(0,t) = 0)$ and $\mathsf{P}(X(t, t + \delta t) = 0)$, using the axioms and assuming the process starts at time 0, show that

$$p_0(t) = e^{-\lambda t}.$$

(iii) Vehicles categorized into two types pass under a bridge according to independent Poisson processes with rates of $\lambda_H = 2$ per minute for heavy goods vehicles and $\lambda_S = 8$ per minute for all other smaller vehicles.

(a) Let N be the total number of vehicles that pass under the bridge in three minutes. Find $\mathsf{E}(N)$ and $\mathsf{var}(N)$.

(b) What is the probability of seeing a heavy goods vehicle before a smaller vehicle?

(c) If, at the end of three minutes you have seen 24 small vehicles, but no heavy goods vehicles, what is the probability you will see exactly one heavy goods vehicle in the next minute?

(d) What is the expected value of the random variable representing the time until you have seen at least one of each type of vehicle?

2. (i) Define the Galton-Watson discrete time branching process.

(ii) In a Galton-Watson discrete time branching process which starts at generation 0 with one individual, use probability generating functions to show that the mean size of generation n can be expressed in terms of the mean size of generation 1.

(iii) Three branching processes have offspring probability functions given by

j	0	1	2	3	4	5	≥ 6
$p_1(j)$	0	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	0
$p_2(j)$	0	0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0
$p_3(j)$	t	$1-2t$	0	t	0	0	0

where $0 < t \leq \frac{1}{2}$.

(a) Find the probability generating functions for the three processes.

(b) Show that p_1 and p_2 have the same first and second moments.

(c) Find the probability of ultimate extinction for the three processes.

3. A and B play a sequence of independent games in which A pays B £1 when B wins and B pays A £2 when A wins. If A has no money and loses, he does not have to pay. B has an infinite amount of money. The probability that A wins any particular game is $p = \frac{9}{19}$. Let $X_n, n = 0, 1, 2 \ldots$ be A's total winnings after n games, and q_j be the probability that A ever has exactly £2 if the game starts with $X_0 = j$.

(i) Find expressions in terms of n and p for $\mathsf{E}(X_n)$ and $\mathsf{var}(X_n)$.

(ii) Derive a recurrence relation for q_j in terms of q_{j-1} and q_{j+2}, valid for $j = 1$ and for $j \geq 3$.

(iii) Show that for $j > 2$, the solution $q_j = \left(\frac{2}{3}\right)^{j-2}$ satisfies the recurrence relation found in part (ii).

(iv) Write down a recurrence relation for q_0 and hence find q_0 and q_1.

4. (i) For a discrete time Markov Chain with transition matrix $(P = p_{ij}; i, j \in S)$, where S is the state space of the chain, explain what is meant by saying that the Chain is
(a) aperiodic and (b) irreducible.

(ii) A discrete time Markov Chain which is in state X_n after n transitions, has state space given by the non-negative integers and transition probabilities

$$p_{ij} = \begin{cases} 0 & j > i+1 \\ \frac{1}{i+2} & 0 \leq j \leq i+1. \end{cases}$$

(a) Determine whether the chain is irreducible and aperiodic.

(b) Prove that the stationary distribution satisfies $\pi_i = \pi_0 / i!,\ i \geq 0$.

(c) Find and name the distribution of X_n as $n \to \infty$.

5. Let $(X(t);\, t \geq 0)$ be a continuous time Markov process with state space $S = \{0, 1, 2, \ldots\}$.

Denote the transition and transition rate matrices by $(P(t) = p_{i,j}(t);\, i, j = 0, 1, 2, \ldots)$ and $(Q = q_{i,j};\, i, j = 0, 1, 2, \ldots)$ respectively.

(i) Write down the relationship between the elements of $P(\delta t)$ and the elements of Q.

(ii) Assume that $X(t)$ is the number of events by time t in a Poisson process with rate λ and $X(0) = 0$, so that, for $t \geq 0$ and $i = 0, 1, 2, \ldots$,

$$\begin{aligned} \mathsf{P}(X(t+\delta t) = i+1 \,|\, X(t) = i) &= \lambda \delta t + o(\delta t) \\ \mathsf{P}(X(t+\delta t) = i \,|\, X(t) = i) &= 1 - \lambda \delta t + o(\delta t) \end{aligned}$$

By considering the first row of the forward equations for this process show that $X(t)$ has a Poisson distribution with parameter λt.

(iii) The lifecycle of a particular organism is categorized into three states as follows: it is born and stays in state A for a time that is exponentially distributed with parameter δ_A then it changes to state B for a time that is exponentially distributed with parameter δ_B then it dies (state D).

(a) Write down the Q matrix for this process.

(b) Write down the relationship between Q and the stationary distribution and hence find the stationary distribution.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	PGF $\Pi_X(s)$
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta s$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta s)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(s-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta s}{1-s(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta s}{1-s(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-s(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	