Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3S4/M4S4

Applied Probability

Date: Thursday, 18th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are included on pages 5 & 6.

1. (i) Write down the three axioms of the Poisson process with rate λ.

(ii) Let $X_1, X_2, \ldots, X_k$ be independent Poisson random variables with means $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. Using probability generating functions show that $X = X_1 + X_2 + \ldots + X_k$ is also a Poisson random variable with mean $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$.

(iii) Let the process $X(t) = X_1(t) + X_2(t) + \ldots + X_k(t)$ be the superposition of k independent Poisson processes, $X_1(t), X_2(t), \ldots, X_k(t)$ with rates $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. Using the axioms of the Poisson process show that

(a) $X(t)$ is also a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$.

(b) Given the occurrence of a point in the process $X(\cdot)$ at time t show that the point was generated from process $X_i(\cdot)$ with probability λ_i/λ $(i = 1, 2, \ldots, k)$.

(iv) My television can be either **on** (state 1) or **off** (state 0). Jack is pressing the off switch on his remote control according to a Poisson process with rate λ_0 independently of Elizabeth, who is pushing the on switch on her remote control according to a Poisson process with rate λ_1. The processes start at time 0 when the television is initially off. Let $Z(t)$ represent the state at time t and $N(t)$ be the total number of times a switch is pressed in $[0, t)$.

Using the results of part (iii) or otherwise, show that, for any $t \geq 0$

(a)

$$\mathsf{P}(N(t) \geq 1) = 1 - e^{-(\lambda_0+\lambda_1)t}.$$

(b)

$$p_{01}(t) = \mathsf{P}(Z(t) = 1 \,|\, Z(0) = 0) = \left(1 - e^{-(\lambda_0+\lambda_1)t}\right) \frac{\lambda_1}{\lambda_0 + \lambda_1}.$$

2. (i) Define the Galton-Watson discrete time branching process.

(ii) A branching process starts with one individual, and has offspring probability function given by

$$p(j) = \left(\frac{1}{2}\right)^{j+1} \qquad j = 0, 1, 2, \ldots$$

(a) Prove by induction that the probability generating function for the size of the nth generation is given by

$$\Pi_n(s) = \frac{n - (n-1)s}{n + 1 - ns}.$$

(b) What is the mean size of generation n?

(c) Find the probability of ultimate extinction.

3. (i) Let $\{X_n, n = 1, 2, 3, \ldots\}$ be an unrestricted simple symmetric random walk. Show that

$$\mathsf{P}(X_{2m} = 0 \,|\, X_0 = 0) = \binom{2m}{m}\left(\frac{1}{2}\right)^{2m}.$$

By using the following approximation

$$m! \approx e^{-m} m^{m+0.5}\sqrt{2\pi},$$

deduce that such a random walk is recurrent.

(ii) Consider a random walk $\{X_n = Z_1 + \ldots + Z_n; n = 1, 2, 3, \ldots\}$, where $\{Z_n; n = 1, 2, 3 \ldots\}$ is a sequence of independent random variables with

$$\mathsf{P}(Z_i = -2) = \frac{1}{3}, \quad \mathsf{P}(Z_i = +2) = \frac{1}{2}, \quad \mathsf{P}(Z_i = +4) = \frac{1}{6}.$$

(a) Find $\mathsf{E}(X_n)$ and $\mathsf{var}(X_n)$.

(b) Find an expression for an estimate of the probability that $X_n \leq x$ when n is large.

4. (i) For a Markov Chain with transition matrix $(P = p_{ij}; i, j \in S)$, where S is the state space of the chain, define

(a) the communicating classes of the Markov chain.

(b) the periodicity of state i.

(ii) To save time spent marking assessed coursework, grades, which can be either A, B or C, are to be based on the student's performance in the previous piece of coursework using the following transition matrix

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.1 & 0.9 \end{pmatrix}$$

(a) Find the stationary distribution associated with this transition matrix. With justification, state whether this is also the limiting distribution.

(b) If your first piece of coursework is assigned a grade B, what is the expected number of further pieces of coursework until you next receive a B grade?

Question 4 is continued on Page 4

(iii) Let $\{X_n;\, n=0,1,2,\ldots\}$ be an irreducible, discrete time Markov chain with $X_0=0$ (i.e. starting at the origin). Let f denote the probability that X eventually returns to the origin, and N be the random variable representing the total number of visits to the origin (where $X_0=0$ counts as a visit).

(a) If $f<1$, show that

$$\mathsf{P}(N=j)=(1-f)f^{j-1}, \quad j=1,2,3,\ldots$$

(b) Let N_n be the indicator random variable for whether X_n is at the origin at time n:

$$N_n=\begin{cases} 1 & X_n=0 \\ 0 & X_n\neq 0. \end{cases}$$

By writing N in terms of the N_n, evaluate $\mathsf{E}(N)$ and hence prove that state 0 is transient if and only if

$$\sum_{n=1}^{\infty}\mathsf{P}(X_n=0\,|\,X_0=0)<\infty.$$

5. Let $(X(t);\, t\geq 0)$ be a continuous time Markov process with state space $S=\{0,1,2,\ldots\}$. Denote the transition and transition rate matrices by $(P(t)=p_{i,j}(t);\, i,j=0,1,2,\ldots)$ and $(Q=q_{i,j};\, i,j=0,1,2,\ldots)$ respectively, with

$$\begin{aligned} p_{n,n+1}(\delta t) &= \lambda\delta t+o(\delta t) && n=0,1,2,3,\ldots \\ p_{n,n-1}(\delta t) &= \nu\delta t+o(\delta t) && n=1,2,3,\ldots \\ p_{i,j}(\delta t) &= o(\delta t) && \text{otherwise} \end{aligned}$$

(i) Write down the form of Q for such a process.

(ii) Write out the forward differential difference equations for $p_{i,j}(t)$ in terms of λ and ν.

(iii) Show that π is the stationary distribution of the process if and only if it satisfies

$$\pi_i q_{i,i+1}=\pi_{i+1}q_{i+1,i}, \quad i\geq 0.$$

(iv) Using part (iii) or otherwise, show that the stationary distribution π exists if and only if $\nu>\lambda$, and is then given by

$$\pi_i=\frac{\nu-\lambda}{\nu}\left(\frac{\lambda}{\nu}\right)^i, \quad i\geq 0.$$

(v) For $\nu>\lambda$, show that as $t\to\infty$,

$$\mathsf{E}(X(t))\to\frac{\lambda}{\nu-\lambda}.$$

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$\left(1-\theta+\theta e^t\right)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\left\{\lambda\left(e^t-1\right)\right\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (std model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (std model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (std model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (std model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (std model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2 t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	