1. (a) (i) Write down the three axioms of the Poisson process with rate λ.

(ii) Using the axioms prove that the probability of obtaining no realizations in the interval $[0, t)$ for a Poisson process with rate λ is $e^{-\lambda t}$.

(iii) For a non-homogeneous Poisson process with rate $\lambda(t) = 1 + \sin(t)$, determine the probability of obtaining no realizations in the interval $[0, t)$

(b) In a particular examination, a student must answer questions from an inexhaustible supply of increasingly difficult questions. In the interval $[t, t + \delta t)$, in minutes, the probability that a student completes more than one question is $o(\delta t)$, and the probability of completing exactly one question is

$$\frac{2}{1+t}\,\delta t + o(\delta t).$$

The examination starts at time zero. Using the deterministic approximation, find an expression for the expected number of questions the student completes by time t.

2. (a) Consider the branching process for which the number of offspring, X, of each individual has $\mathsf{E}(X) = 1$ and var$(X) = \sigma^2$. Let Z_n be the number of individuals in generation n. Assuming $Z_0 = 1$, use probability generating functions to show that,

(i) $\mathsf{E}(Z_n) = 1$,

(ii) var$(Z_n) = n\sigma^2$.

(b) Now consider the branching process for which an individual has 0 offspring with probability α, or 3 offspring with probability $1 - \alpha$.

(i) For what values of α is extinction not certain?

(ii) When $\alpha = 1/2$ find the probability of extinction.

3. (a) A particle performs a random walk on the integers, moving 2 units to the right with probability p and 1 unit to the left with probability $q = 1 - p$.
Define A_i to be the probability that the particle ever visits the origin, given that it starts from position i.

(i) Show that

$$A_i = (A_1)^i,$$

(ii) and hence show that A_1 satisfies the following cubic equation,

$$pA_1^3 - A_1 + q = 0.$$

(iii) Given that $x = 1$ is one solution to the cubic equation $px^3 - x + q = 0$, solve the equation in part (ii) to find the probability that the particle ever visits the origin given that it starts in position 1, distinguishing between the two cases $p \leq 1/3$ and $p > 1/3$.

(b) On any day the value of a share may increase by £0.40 with probability 0.5, may remain unchanged with probability 0.2 or may decrease by £0.20 with probability 0.3. Initially the share has value £100.00. Let X_t be the value of the share after t days, where X_t may be negative.

(i) Write down expressions for $\mu_t = \mathsf{E}(X_t)$ and $\sigma_t^2 = \mathsf{var}(X_t)$.

(ii) Obtain a normal approximation to the value of the share in the long run.

(iii) Estimate the probability that after 365 days the share is worth at least £110.

(leaving your answers for parts (ii) and (iii) in terms of μ_t, σ_t^2 and $\Phi(\cdot)$ – the cdf of the standard normal distribution)

4. (a) What is meant by saying that a Markov chain is

   (i) irreducible?

   (ii) aperiodic?

   (b) Two urns contain a total of three balls. At each step one of the three balls is chosen at random (each with equal probability) and transferred to the other urn. Initially all balls are in the second urn, and the subsequent state of the system is described by the number of balls in the first urn.

   (i) What is the transition matrix for the system?

   (ii) Draw the corresponding transition diagram.

   (iii) Does a unique stationary distribution exist?

   (iv) What condition is needed for a stationary distribution to also be a limiting distribution? Is this condition met for this system?

   (v) Find both the stationary distribution and the expected number of steps until the first urn is empty again.

5. (a) For a continuous time, discrete state space Markov process with transition matrix $P(t)$,

   (i) write down the corresponding forward differential equations.

   (ii) write down the corresponding equations satisfying the stationary distribution $\boldsymbol{\pi}$, if it exists.

   (b) A lecturer's mood varies according to a continuous time Markov process with two states, good or bad. When her mood is good, the probability of a change of her mood in the interval $[t, t + \delta t)$ is $\alpha\, \delta t + o(\delta t)$, and the corresponding probability of a transition from bad to good is $\beta\, \delta t + o(\delta t)$.

   (i) What is the Q-matrix for this process?

   (ii) Use induction to show that $Q^n = (-\alpha - \beta)^{n-1} Q$ and hence show that

$$\exp(tQ) = I + \frac{Q}{\alpha + \beta} + \frac{Q}{-\alpha - \beta} \exp(t(-\alpha - \beta)).$$

   (Recall by definition that $\exp(tQ) = I + \sum_{n=1}^{\infty} \frac{(tQ)^n}{n!}$, where I is the identity matrix).

   (iii) Let $P(t) = \exp(tQ)$. Verify that the expression for $P(t)$ found in part (ii) is satisfied by the backward differential equations.

   (iv) Find the stationary distribution associated with this process.