UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3S4 APPLIED PROBABILITY

Date: Wednesday, 9th June 2004 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (a) In a continuous time process, the occurrence of events may be modelled by the following axioms:

1. P(exactly one event occurs in any small interval of length h) $= \lambda h + o(h)$.
2. P(two or more events occur in any small interval of length h) $= o(h)$.
3. The occurrence of events after time t is independent of the occurrence of events before time t.

(i) Using these axioms, write down an expression for $p_n(t+h)$, the probability that n events will have occurred by time $t+h$, in terms of the number of events which have occurred in the intervals $[0,t)$ and $[t,t+h]$.

(ii) From this derive the differential difference equations for this process.

(iii) Hence derive a differential equation for the probability generating function of the process.

(iv) For a given time $t > 0$, solve this equation, given that at time 0 no events have occurred.

(v) Name the distribution which has the probability generating function given in part **(iv)**.

(b) Suppose that road accidents follow a Poisson process with rate λ and that Y_i, the number of casualties in the ith accident $(i = 1, 2, \ldots)$, has a geometric distribution $Y_i \sim G_0(q)$, what is the mean of the number of casualties which have occurred by time t? You may use, without proof, any results you know about the moments of the geometric distribution.

2. The lifetimes of individuals in a certain population are independent and follow an exponential distribution distribution with parameter ν. During their lifetimes, individuals in the population independently give birth to offspring according to a Poisson process with rate β.

(a) Derive and name the distribution of the total number of offspring *born directly to* a single individual.

(b) Derive the mean and probability generating function of the distribution obtained in **(a)**.

(c) Hence derive the probability that the population will die out, in terms of β and ν, if it starts with a single member. You may use any general standard results about the probability that a population will die out.

(d) If $\beta \leq \nu$, and the process starts with a single individual, derive the expected total number of births. You may use any result you know about the expected number of births in the nth generation in terms of the mean of the offspring probability distribution.

3. **(a)** **(i)** Let P be the transition probability matrix of a Markov chain and let π be a probability distribution over its states. What constraints must π satisfy in order for it to be a stationary distribution?

(ii) What is the characteristic property of a closed class in a Markov chain?

(iii) Define the term 'communicating class' and prove that the states of a finite Markov chain are partitioned into non-overlapping communicating classes.

(b) Many medical situations can be modelled as Markov chains: individuals fall ill and then move through progressively more serious or less serious disease states until they die or recover. One such disease, involving ruptured blood vessels in the brain, has four states $0, 1, 2$, and 3. The matrix, P, of transition probabilities between the states is given below

$$P = \begin{bmatrix} 0.5 & 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.125 & 0.125 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Draw the transition diagram of this matrix.

(ii) List the communicating classes of this transition matrix, and say which of them are closed.

(iii) Either prove that the matrix has a unique stationary distribution, or give two different stationary distributions.

(c) Medical research leads to the discovery that if the brain is cooled rapidly after damage has occurred, then the transition matrix given in part **(b)** is changed to Q, given below.

$$Q = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

(i) Find the stationary distribution of this transition matrix.

(ii) What is the mean recurrence time of state 0.

4. **(a)** Define the traffic intensity of a queue and write down the traffic intensity of an $M(\lambda)/M(\nu)/1$ queue.

(b) For an $M(\lambda)/M(\nu)/1$ queue, the differential difference equations for the probability that there will be x objects in the queue at time t are

$$\frac{d}{dt}p_x(t) = \lambda p_{x-1}(t) + \nu p_{x+1}(t) - (\lambda + \nu)p_x(t) \quad \text{for } x = 1, 2 \ldots$$

and

$$\frac{d}{dt}p_0(t) = \nu p_1(t) - \lambda p_0(t).$$

Assuming the distributions $p_x(t), x = 0, 1, 2, \ldots$ settles down to a steady state distribution $p_x, x = 0, 1, 2, \ldots$, use these equations to find p_1 in terms of p_0, and p_2 in terms of p_0, and then as many other terms as are necessary for you to decide on the general form for p_x in terms of p_0. Write down this general form.

(c) Using the general form for p_x obtained in **(b)**, determine a condition, in terms of the traffic intensity ρ, for the queue length to have a steady state distribution. What is this distribution?

(d) Observations of the queue are taken at widely spaced random times. At each observation, the queue length and the time to service completion for the person currently being served is noted. From these figures, the mean queue length is found to be a, and the mean time to service completion for the person currently being served is found to be b. What is the mean inter-arrival time of the customers.

5. (a) A particular stochastic process leads to the following partial differential equation for its probability generating function $\Pi(s,t)$:

$$2st\frac{\partial\Pi(s,t)}{\partial s} - s\frac{\partial\Pi(s,t)}{\partial t} = 2t\Pi(s,t).$$

Using the auxiliary equation relating ds to dt and the equation relating ds to $d\Pi$, solve this equation for $\Pi(s,t)$ when it is known that $\Pi(s,0) = se^s$.

(b) A bacterial infection of an organism reproduces according to a simple birth process with rate β. Initially, there is only one bacterium infecting the organism. The organism is treated with a drug at random times following a Poisson process with rate γ. Whenever the organism is treated, all of the bacteria are killed, and once this has happened no bacteria reappear.

(i) Write down the probability $\mathsf{P}(Y(t) > 0)$, where $Y(t)$ is the size of the bacterial population alive at time t.

(ii) The probability generating function of the conditional probability distribution $p_y = P(Y(t) = y \,|\, Y(t) > 0), y = 1, 2, \ldots$ is

$$\Pi(s,t) = \frac{se^{-\beta t}}{1 - s(1 - e^{-\beta t})}$$

What is the probability mass function corresponding to this conditional probability?

(iii) Use your answers to parts **(i)** and **(ii)** to write down the unconditional probability distribution of the bacterial population size $Y(t)$ at time t. Hence find the probability generating function $\Pi(s,t)$ for $Y(t)$.

(iv) Hence calculate the mean bacterial population size at time t.