1. Consider the regression model

$$Y_t = \beta_0 + \beta_1 X_{t-1} + \epsilon_t, \quad t = 0, \ldots, N-1,$$

where X_t is a random process, and ϵ_t are independently identically distributed $N(0, \sigma^2)$, so that

$$\text{cov}(\epsilon_t, \epsilon_s) = \sigma^2 \delta_{t,s},$$

where

$$\delta_{t,s} = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

Assume that the joint distribution of $\boldsymbol{Y} = (Y_0, \ldots, Y_{N-1})^\top$ and $\boldsymbol{X} = (X_0, \ldots, X_{N-1})^\top$ depends on the full set of parameters $\boldsymbol{\lambda} = (\boldsymbol{\beta}, \boldsymbol{\beta}')$, where $\boldsymbol{\beta} = (\beta_0, \beta_1)^\top$, and $\boldsymbol{\beta}'$ is a vector of additional parameters that appear in the joint distribution.

(a) Write down the definition of the statement that "X_t is weakly exogenous for $\boldsymbol{\beta}$".

(b) Prove that if X_t is weakly exogenous for $\boldsymbol{\beta}$, then inference on $\boldsymbol{\beta}$ can be based on the conditional distribution of Y_t, that is

$$f(y|\boldsymbol{X}, \boldsymbol{\beta}) = \prod_{t=0}^{N-1} f_{Y_t|Y_{t-1},X_t}\left(y_t|\boldsymbol{X}_t, Y_{t-1}; \boldsymbol{\beta}\right) \quad (1)$$

(c) Assume now that

$$X_t = \theta X_{t-1} + \eta_t,$$

where η_t are independently identically distributed $N(0, \sigma^2)$, and

$$\text{cov}\left(\epsilon_t, \eta_s\right) = \rho\sigma^2 \delta_{t,s},$$

where it is assumed that σ^2 and ρ are known. By rewriting the equation for Y_t in the form

$$Y_t = \beta_0 + \gamma_1 X_t + \gamma_2 X_{t-1} + \epsilon'_t,$$

or otherwise, show that X_t is exogenous for $[\beta_0,\ \beta_1 - \rho\theta]^\top$.

2. The rate of return to speculation k time points ahead of the current time point t, rr_{t+k} can be modelled in terms of the logarithm of the spot exchange rate, s_t, and $g_{t,k}$, the logarithm of the k period forward rate, if set at time point t. We assume that the rate of return depends on $t+k$ only. Specifically, suppose that

$$rr_{t+k} = s_{t+k} - g_{t,k}. \tag{2}$$

(a) Denote by J_t all the information available at time t. The efficient market hypothesis corresponds to the fact that *"prices fully reflect all available information."* By considering $E(\cdot|J_t)$, or otherwise, state what the efficient market hypothesis corresponds to in (2).

(b) In order to test the efficient market hypothesis, a model for rr_{t+k} of the form

$$rr_{t+k} = s_{t+k} - g_{t,k} = \alpha_0 + \alpha_1 rr_t + \alpha_2 rr_{t-1} + \epsilon_{t+k}.$$

is to be used. Find $E(rr_{t+k}|J_t)$, based on the assumption $E(\epsilon_{t+k}|J_t) = 0$. Explain how to state the efficient market hypothesis in terms of this model.

(c) Assume that ϵ_t, s_{t+k} and $g_{t,k}$ are mixing, so that subsequent estimation procedures are consistent. Is it reasonable to assume

$$E(\epsilon_{t+k}|\epsilon_{t+k-j}) = 0 \qquad \text{for all } j \in \mathbb{Z}?$$

Justify your answer carefully.

(d) Parameters $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$, that relate the systematic variation in the rate of return to speculation to previous rates of return, are to be estimated using least squares, based on a sample of size N of observed rates of return. Based on your answer to (c) write down the model for the covariance structure of ϵ_t, with justification of why this model is appropriate.

(e) Denote an estimate of $\boldsymbol{\alpha}$ by $\widehat{\boldsymbol{\alpha}}$, and an estimate of var $(\widehat{\boldsymbol{\alpha}})$ by $\widehat{\boldsymbol{D}}$. Describe briefly how to test formally the efficient market hypothesis using the Wald statistic.

3. Consider two related time series, one, X_t, corresponding to the price of crude oil per gallon, the second, Y_t, corresponding to the Gross Domestic Product (GDP) of the US.

It is generally assumed that the behaviour of X_t and Y_t are linked, in that one of the two series causes the other.

(a) Give the definition of the statement that *"X Granger causes Y."*

(b) The joint behaviour of X_t and Y_t is to be modelled via

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha_3 & \alpha_4 \\ \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} X_{t-2} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix},$$

where $\boldsymbol{\epsilon}_t = [\epsilon_{1t}, \epsilon_{2t}]^\top$ is assumed to be an innovation term.
In terms of the parameters of the model, give the conditions that must hold if X Granger causes Y.

(c) Are there any reasons that, in practice, your answer to (b) would not be sufficient to assert Granger causality for two real data series? Explain your reasoning carefully.

(d) Given realisations $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ from X_t and Y_t respectively, describe how to estimate the parameters of the model using a dynamic regression model with design matrix $\mathcal{X}$, and least squares, under the assumption that $\boldsymbol{\epsilon}_t = [\epsilon_{1t}, \epsilon_{2t}]^\top \sim N(0, \sigma_\epsilon^2 \boldsymbol{I})$, independently across t. *(Do NOT carry out the matrix calculations.)*

(e) Let $\boldsymbol{\epsilon}_1 = [\epsilon_{13}, \ldots, \epsilon_{1N}]^\top$ and $\boldsymbol{\epsilon}_2 = [\epsilon_{23}, \ldots, \epsilon_{2N}]^\top$ and construct the vector

$$\boldsymbol{\epsilon} = \begin{bmatrix} \boldsymbol{\epsilon}_2 \\ \boldsymbol{\epsilon}_2 \end{bmatrix}.$$

Under the assumption that we have $\widehat{\Omega}_N$, a non-singular estimator of the covariance of $\widehat{\gamma}$, where $\widehat{\gamma}$ is a parameter vector to be specified, construct a Wald test for a suitable null hypothesis to clarify any potential causal relationship using the model in (b).

4. Consider observing three time series corresponding to $X_t^{(1)}$, the FTSE 100 and $X_t^{(2)}$ and $X_t^{(3)}$, two individual company stock prices.

(a) Assume that a structural model

$$X_t^{(i)} = \alpha_i \mu_t^{(i)} + \epsilon_t^{(i)}, \quad i = 1, 2, 3,$$

is to be used , where $\mu_t^{(i)}$ represents the trend component (growth) of series i, whilst $\epsilon_t^{(i)}$ is the stochastic (irregular) component of series i.

Assume the following,

$\mu_t^{(i)} \sim I(1)$.
$\epsilon_t^{(i)} \sim I(0)$, are three mutually independent stationary processes.
$\mu_t^{(i)}$ is independent of $\epsilon_t^{(j)}$ for all i and j.
$\mu_t^{(i)}$ and $\mu_t^{(j)}$ may be dependent.

Recall that a process with a unit root is denoted $I(1)$.

(i) If $\mu_t^{(i)}$, $i = 1, 2, 3$ are three *independent* processes, determine whether

$$Y_t = \frac{X_t^{(2)}}{\alpha_2} + \frac{X_t^{(3)}}{\alpha_3} - \frac{2X_t^{(1)}}{\alpha_1}$$

is $I(0)$ or $I(1)$.

(ii) If $\mu_t^{(i)} = \mu_t$, $i = 1, 2, 3$ are all equal to the *same* process, determine whether

$$Y_t = \frac{X_t^{(2)}}{\alpha_2} + \frac{X_t^{(3)}}{\alpha_3} - \frac{2X_t^{(1)}}{\alpha_1}$$

is $I(0)$ or $I(1)$. How would this assumption be interpreted?

(iii) If the assumptions of (ii) are still valid, describe how to estimate $\boldsymbol{a} = \left[\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}\right]^\top$, with the normalisation $\alpha_1^{-1} = 1$.

(b) Suppose that

$$X_t = \phi X_{t-1} + \epsilon_t,$$

where ϵ_t is assumed to be identically and independently distributed. Describe how to test the hypothesis that $\phi = 1$, based on a sample of length N.

5. Consider a credit portfolio as a collection of m transactions. Each of the m transactions corresponds to the party defaulting, or not defaulting. Let $L_i = 1$ if obligator i defaults, and 0 otherwise. Consider the vector of random variables $\boldsymbol{L} = (L_1, \ldots, L_m)^\top$. Let $\boldsymbol{p} = (p_1, \ldots, p_m)^\top$ be the default probabilities for $\boldsymbol{L}$, i.e. $p_i = \mathsf{P}(L_i = 1)$, where $\boldsymbol{p}$ is modelled as a random vector, and consider the Bernoulli loss model where $(p_1, \ldots, p_m)$ follow some distribution on $[0,1]^m$, and conditional on $(p_1, \ldots, p_m)$, the L_i are independent Bernoulli random variables.

Denote the loss, K, of $\boldsymbol{L}$ as

$$K = \sum_{i=1}^{m} L_i.$$

(a) Suppose that p_i all take the *same* value P which is uniformly distributed on $[0,1]$ i.e. $P \sim U(0,1)$.

(i) Find $\mathsf{P}\left(L_1 = l_1, \ldots, L_m = l_m\right)$, where $l_i \in \{0,1\}$ for $i = 1, \ldots, m$.

(ii) Find the probability that k defaults occur where $0 \leq k \leq m$, i.e. find $\mathsf{P}(K = k)$.

(b) Suppose now that, $\boldsymbol{P} = (P_1, \ldots, P_m)^\top$ and $\boldsymbol{P} \sim F(\boldsymbol{p}) = \mathsf{P}(P_1 \leq p_1, \ldots, P_m \leq p_m)$, with $\mathsf{E}\left(\boldsymbol{P}\right) = \boldsymbol{\mu_P}$ and $\mathrm{var}\left(\boldsymbol{P}\right) = \boldsymbol{\Sigma_P}$.

(i) Find the (marginal) covariance of $\boldsymbol{L}$.

(ii) Determine when the covariance take its minimum and maximum possible values. Are either of those two scenarios realistic practically ?

(iii) Explain how the variance of $\boldsymbol{P}$ affects the default covariance in (b)(i).