Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2007

This paper is also taken for the relevant examination for the Associateship.

M3S15/M4S15

Monte Carlo Methods in Financial Engineering

Date: Monday, 21 May 2007 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (a) Suppose we can sample a batch $X_1, X_2, \ldots, X_n$ of i.i.d. random variables from a given distribution, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$. Show that the sample mean and sample variance

$$\widehat{\mu_n} := \bar{X}_n := \frac{1}{n}\sum_1^n X_i \quad \text{and} \quad \widehat{\sigma}_n^2 := \frac{1}{n-1}\sum_1^n (X_i - \bar{X}_n)^2$$

are unbiased estimators for the mean μ and the variance σ^2, respectively. That is, show that $E[\widehat{\mu_n}] = \mu$ and $E[\widehat{\sigma}_n^2] = \sigma^2$.

(b) Compute the variance and the standard deviation (standard error) of $\widehat{\mu_n}$. Suppose that a first sample of size n and has yielded an estimate for the standard deviation of $\widehat{\mu_n}$ of 0.20. How big would you choose the size m for a second independent sample batch in order to achieve, approximately, a standard deviation of $\frac{1}{100}0.20$ for the sample mean?

(c) In the Black Scholes model the stock price at time T is

$$S_T = s \, \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T \right)$$

where $s, r, \sigma > 0$ are the fixed usual parameters for the initial stock price, interest rate, and volatility, and (W_t) is a Brownian motion under the risk-neutral measure P.

Suppose you can simulate i.i.d. standard normal random variables $Z_1, Z_2, \ldots$. Describe how one can compute by Monte Carlo simulation the price $E[\exp(-rT)h(S_T)]$ of a European put option with strike $K > 0$ that delivers the payoff $h(S_T) = (K - S_T)^+$ at maturity time T.

2. Let $(S_t)_{t \geq 0}$ be the price process of a stock that pays no dividends, and let r denote the riskfree interest rate for both borrowing and lending.

(a) (i) Explain in words what a forward contract is, given the stock is the underlying; this should include a careful explanation of all terminology that is involved.

(ii) Supposing the forward contract is settled in cash, what is the payoff at maturity T for a short position in the forward contract whose delivery price F_t was agreed at time $t < T$? What is the profit from entering a short forward position at time t and holding it until T ? Sketch both the payoff and the profit of a short forward position at time T as a function of S_T.

(b) Demonstrate that there is a unique no-arbitrage forward price F_t, in the sense that any other forward price would imply that there exists arbitrage opportunities.

(c) Consider two forward contracts on the same underlying stock with the same maturity date T, that have been entered at different time $t_1 < t_2 < T$. What is the value of a short position of the first contract at the time t_2 when the second contract is entered?

Hint: First, calculate the difference between the payoffs for short positions in both forward contracts at maturity.

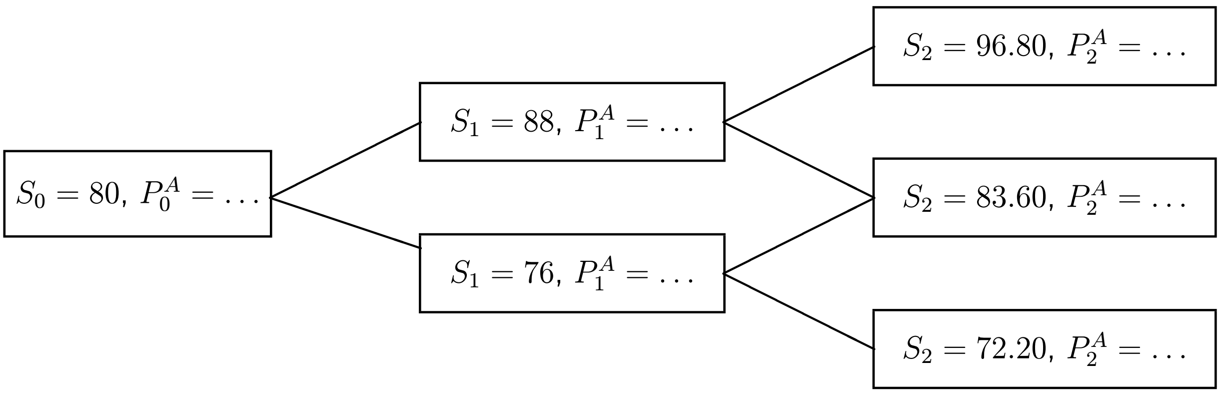
3. Consider the binomial model for a financial market on a finite probability space $(\Omega, \mathcal{F}, P)$ with $T = 2$ periods of length $T/2 = 1$. Let Y_1, Y_2 be i.i.d. random variables under the real world probability $\widetilde{P}$ with $\widetilde{p} := \widetilde{P}[Y_k = 1.1] = 1 - \widetilde{P}[Y_k = 0.95] = \frac{1}{2}$ and let the stock price process S be given by $S_0 = 80$ and $S_k = S_{k-1}Y_k$ for $k = 1, 2$. The interest rate r is such that $\exp(r) = 1.05$. The filtration $(\mathcal{F}_k)_{k=0,1,2}$ modelling the information flow is given by $\mathcal{F}_k = \sigma(Y_i \,|\, i \le k) = \sigma(S_i \,|\, i \le k)$.

(a) Derive the risk neutral transition probability p for an 'up'-move (and $1 - p$ for 'down') of the risk neutral measure P from the equation

$$E^P\left[\frac{S_{k+1}}{S_k}\,\middle|\,\mathcal{F}_k\right] = \exp(r)\,.$$

Explain how this equality motivates the name of the "risk neutral measure".

(b) Compute the price (P_k^A) of the American put option with strike $K = 80$ on the stock at all nodes of the binomial tree. Explain the way of computation, and record your results in a sketched diagram of the tree, similar to the diagram below that details which prices the stock (S_k) is taking on each tree node.



(c) State at which of the nodes at times $k = 0$ and $k = 1$ the American put option would be exercised under optimal exercise policy, and how the decision between exercising the option and continuation is made at a given node.

4. For $T > 0$ and $n \in \mathbb{N}$, let $t_i := i\frac{T}{n}$ for $i = 0, 1, \dots, n$.

(a) In the Black Scholes model, the stock price under the risk neutral measure is given by

$$S_t = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma W_t)$$

where $S_0, r, \sigma > 0$ denote the usual parameters for the initial stock price, interest rate, and volatility, and where W denotes a Brownian motion.

Derive the stochastic differential equation SDE for the process S_t and for the process $X_t := \log(S_t)$. Then write down the Euler scheme for a numerical approximation of the SDE for (X_t). Then show that the scheme is actually exact for (X_t), in that it does not cause an approximation error for the discretely sampled path of (X_t).

(b) Now consider the process $Y_t := \cos(W_t)$. Derive the SDE for (Y_t) and write down the Euler scheme for the numerical approximation of the SDE.

Explain in detail how you can, based solely on a sequence of i.i.d. standard normal random variables $Z_1, Z_2, \dots$, simulate a discretely sampled path $(Y_{t_i})_{i=0,1,\dots,n}$

(i) approximatively by applying the Euler scheme discretisation to the SDE of Y;

(ii) exactly without using the Euler scheme approximation to the SDE of Y.

(c) In the stochastic volatility model by Heston, the instantenous volatility fluctuates itself randomly, and could have correlation with the stock price returns. More precisely, the joint evolution of the stock price (S_t) and the volatility $(\sqrt{V_t})$ under the risk neutral measure is described by the 2-dimensional SDE

$$dS_t = S_t r\,dt + S_t\sqrt{V_t}\,dW_t^S\,, \quad S_0 = s > 0, \tag{1}$$
$$dV_t = \kappa(\bar{v} - V_t)\,dt + \eta\sqrt{V_t}\,dW_t^V\,, \quad V_0 = v > 0\,, \tag{2}$$

where $r, \kappa, \bar{v}, \eta$ are strictly positive parameters, and the two Brownian motions are correlated with parameter $\rho \in (-1, 1)$; that means

$$\begin{pmatrix} dW_t^S \\ dW_t^V \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt\right).$$

Suppose that you want to simulate the joint evolution of the process (X_t, V_t) for $X_t := \log S_t$ along the discrete time grid $t_0, t_1, \dots, t_n$ based solely on a sequence of i.i.d. standard normal random variables $Z_1, Z_2, \dots$.

(i) Write down the standard Euler scheme for a numerical approximation for the (joint) SDE of X_t and V_t, and explain the problem that occurs.

(i) Suggest a modification of the standard Euler scheme to handle the problem and simulate paths of (X_t, V_t).

5. We consider the pricing of multi-asset derivatives, whose payouts depend on the joint evolution of a basket of underlying stocks. Let $(S_t) := (S_t^i)_{t\geq 0}$, $i = 1, \dots, d$ denote the price processes of d stocks. Each S^i is taken to satisfy the stochastic differential equation of geometric Brownian motion $dS_t^i = S_t^i(r\,dt + \sigma^i\,dW_t^i)$ for $t \geq 0$ with $S_0^i = s^i \in (0, \infty)$, under the risk neutral measure, such that the individual stock price processes are

$$S_t^i = S_0^i \exp\left((r - \frac{(\sigma^i)^2}{2})t + \sigma^i\,W_t^i\right)$$

where $s^i > 0$ is the initial price of the ith stock, $r \geq 0$ is the riskless interest rate and $\sigma^i > 0$ is the volatility of the i-th stock and $(W_t) = (W_t^1, \dots, W_t^d)$, $t \geq 0$, is a multivariate Brownian motion such that increments $W_t - W_s$, $t > s$, are independent over disjoint time periods, and are multivariate normally distributed with mean vector zero and $d \times d$ covariance matrix $(t-s)\Sigma$ for $t > s$, with correlation parameter $\rho \in [0, 1)$ and

$$\Sigma := \begin{pmatrix} 1 & \rho & \cdots\cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}.$$

Let $T > 0$, $n \in \mathbb{N}$ and let $t_k := k\frac{T}{n}$ for $k = 0, 1, \dots, n$.

(a) Explain how one can simulate N discretely sampled (joint) price paths $(S_{t_k})_{k=0,1,\dots,n}$ of the stock price processes, based solely on a sequence of i.i.d. univariate standard normal random variables Z_i, $i = 1, 2, \dots$.

(b) Explain how you would choose n and apply (a) to compute the Monte Carlo estimate $\widehat{I}$, based on N simulated paths, for the price $I := E[e^{-rT}H]$ at time 0 of a European call option on the maximum of the d stocks with strike $K > 0$, whose cash-settlement payoff at maturity time T is $H := (\max\{S_T^i \mid i = 1, \dots, d\} - K)^+$.

(c) Next, explain how you would choose n and apply (a) to compute the Monte Carlo estimate $\widehat{I}$, based on N simulated paths, for the price $I := E[e^{-rT}H]$ at time 0 of an Asian call option on the arithmetic (discrete) time average of the average stock prices, whose cash-settlement payoff at maturity time T is

$$H := \left(\left(\frac{1}{365}\sum_{k=1}^{365}\frac{1}{d}\sum_{i=1}^{d} S^i_{\frac{kT}{365}}\right) - K\right)^+ .$$