Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M3S11/M4S11

Games, Risks and Decisions

Date: Friday, 1st June Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) In a two-person zero-sum game the gain to player A when A plays pure strategy a_i and B plays pure strategy b_j is the (i, j)entry of the pay-off matrix G.
In such a game define an inadmissible strategy for player B.

(ii) If $G = \begin{pmatrix} 0 & 4 & 3 \\ 3 & 0 & 2 \end{pmatrix}$ show that b_3 is inadmissible and hence solve the game.

(iii) If $G = \begin{pmatrix} 0 & 4 & 3 \\ 3 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ show that b_3 is inadmissible and hence solve the game.
You should explain your reasoning fully.

2. (i) In a two-person zero-sum game the gain to player A when A plays randomised strategy α and player B plays randomised strategy β is $g(\alpha, \beta)$.
Explain what is meant by (a) an equaliser strategy for player A and (b) a maximin strategy for player A in such a game.

(ii) Show that if A and B have equaliser strategies $\alpha^*$and β^* respectively then α^* is maximin for player A.

(iii) Player B chooses an element from the set of integers $\{2, 3, 4\}$. Player A chooses a subset of two elements from the same set. If A's choices includes B's choice then A wins (and B loses) the amount represented by B's choice. If A's choices do not include B's choice then there is no gain or loss to either side.
Construct the 3 x 3 pay-off matrix.

(iv) Find equaliser strategies for A and B and the value of the game.

3. (i) In a two-person game define (a) an equilibrium pair of strategies and (b) a jointly admissible pair of strategies.

(ii) Give an example to show that it is possible for a pair of strategies to be in equilibrium but not be jointly admissible

(iii) Two opponents A and B are to play a game which consists of two independent rounds. To win the game a player must win both rounds otherwise the game is declared a draw. Each player must decide in advance, without consulting the other, whether to play aggressively (strategy 1) or defensively (strategy 2) and must play the same way in both rounds. If both players play aggressively or if both players play defensively then each player is equally likely to win a particular round. When playing aggressively against a defensive opponent, player A wins (and B loses) a round with probability 0.7 but when playing defensively against an aggressive opponent, A wins (and B loses) a round with probability 0.6.

Assuming that each player wishes to maximise their probability of winning, construct the pay-off table.

(iv) Find a pair of strategies in equilibrium.

(v) Without further calculation sketch the pay-off set and identify the pay-offs corresponding to jointly admissible strategies.

4. (i) Define a utility function.

(ii) Prove that if consequences $C, C^{'}$ and $C^{''}$ are in increasing order of (strict) preference for a decision maker D who is indifferent between $C^{'}$ and the lottery

$$((1-\alpha)C,\ \alpha C^{''})$$

for some $\alpha \in (0,1)$ then

$$u(C^{'}) = a\alpha + b$$

for some constants a and b where $a > 0$, and u is the relevant utility function.

(iii) For $i = 0, 1, 2, 3, ...$, the consequence C_i represents $100i$ pounds. For $i = 1, 2, 3, ...$, decision maker D is indifferent between

$$C_i \quad \text{and} \quad \left(\frac{1}{3}C_{i-1}, \frac{2}{3}C_{i+1}\right).$$

If $u(C_0) = 0$ and $u(C_1) = \frac{1}{2}$ show that $u(C_i) = 1 - (\frac{1}{2})^i$ $(i = 0, 1, 2, ...)$.

(iv) For $n \geq 1$ consider the lottery

$$L_n = (\frac{1}{n}C_1,\ \frac{1}{n}C_2,\ ...,\ \frac{1}{n}C_n),$$

where C_i is defined in (iii).
Show that L_n has utility

$$1 - \frac{1}{n}u(C_n).$$

(v) If lottery $L_n^{'}$ consists of two independent plays of lottery L_n show that

$$u(L_n^{'}) = 1 - \frac{1}{n^2}u(C_n)^2.$$

5. You are driving along a one-way street towards a restaurant where you intend to eat. You know that there will be just 4 possible car parks, 1, 2, 3 and 4 in that order along the road towards the restaurant. Car park 1 is a long way from the restaurant but always has spaces. Car park 2 is closer to the restaurant but only has spaces with known probability $\alpha (0 < \alpha < 1)$. Car park 3 is even closer to the restaurant but it too only has spaces with probability α, independently of car park 2. If these three car parks are full you must park in car park 4 which is right next to the restaurant but this is very expensive although it always has spaces. The utilities to you of the car parks 1, 2, 3 and 4 are respectively 2, 3, 5 and 1 units .

Construct a decision tree to represent this situation and, by considering the cases $\alpha \leqslant \frac{1}{2}$ and $\alpha > \frac{1}{2}$ separately, determine your optimal parking strategy.