Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3S11/M4S11

Games, Risks and Decisions

Date: Friday, 19th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are included on pages 5 & 6.

1. (i) In the context of a zero-sum game between two players A and B, define an *equaliser strategy*.

(ii) Solve the two-person zero-sum games with the following pay-off matrices. The (i,j) entry in each case is the gain to player A when A plays pure strategy a_i and B plays pure strategy b_j.

(a) $\begin{pmatrix} 1 & 3 & 5 \\ 5 & 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ y & y & x \end{pmatrix}$, where x and y are positive real numbers.

2. (i) Define the *lower and upper values*, v_L and v_U respectively, of a two-person zero-sum game in which the gain to player A when A plays randomised strategy α and player B plays randomised strategy β is $g(\alpha, \beta)$.

(ii) Show that if $v_L = v_U$ and if there exist randomised strategies α^* and β^* for A and B respectively such that

$$g(a, \beta^*) \leq g(\alpha^*, b)$$

for all pure strategies a and b for A and B repectively, then

$$\sup_{\alpha} g(\alpha, \beta^*) \;\leq\; g(\alpha^*, \beta^*) \;\leq\; \inf_{\beta} g(\alpha^*, \beta).$$

(iii) Deduce that

$$v_L = v_U = g(\alpha^*, \beta^*)$$

and that α^* and β^* are maximin and minimax strategies for A and B respectively.

You may assume that $\sup_{\alpha} g(\alpha, \beta) = \sup_{a} g(a, \beta)$ for all randomised strategies β for B.

(iv) Without telling player A, player B chooses a cell of a 3×3 square array. Player A can then try to guess which of the 9 cells B has chosen, by naming two adjacent cells in either the same row or in the same column of the array. If either of these two cells is a correct guess then A wins a point and B loses a point. Otherwise neither wins or loses any points.

Let α_p^* denote the randomised strategy for A which gives probability $p\,(0 \leq p \leq \frac{1}{8})$ to each pair of cells which do not include the middle cell of the array and probability $q = \frac{1-8p}{4}$ to each pair of cells which do include the middle cell. Let β^* denote the randomised strategy for B which gives probability one-fifth to each of the four corner cells and to the middle cell of the array.

By finding suitable values of p and q, solve the game.

3. (i) In a cooperative two-person game define (a) *a jointly inadmissible pair of strategies* and (b) *an equilibrium pair of strategies*, for the two players, A and B.

(ii) Two builders, A and B, are competing for a contract to construct a new office block. Each can bid either £L million or £H million for the job, where $L < H$. The builder with the lower bid will win the contract and will be paid the value of their bid once the building is finished. If both bids are equal, a fair coin is tossed to decide who should win the contract. Each builder reckons that the real cost of completing the job is £c million, where $0 < c < 2L - H$. They decide to collaborate.

(a) Construct the table of pay-offs to the two builders and describe the pay-off set.

(b) Identify the Pareto Optimal set.

(c) Find a pair of strategies in equilibrium.

(d) Find the builders' security levels and identify the negotiation set.

(e) Show that each builder cannot expect to make a profit of more than £m million, where

$$m = \frac{(L-c)(3H-2L-c)}{2(H-c)}.$$

4. (i) Describe, with reasons, the features of the function

$$u(z) = 1 - \exp(-\lambda z) \qquad (z \geq 0),$$

where λ is a positive constant, which might make it suitable to represent the utility of money (£z). How would you interpret the constant λ?

(ii) An individual A with assets of £M ($M > 0$) has the above utility function $u(z)$ for money. If A is offered a lottery L which gives a reward of £R with probability p and a loss of £R with probability $1-p$, where $R < M$ and $0 < p < 1$, show that A will find L attractive provided

$$\lambda \leq \frac{1}{R} \log \frac{p}{1-p}.$$

(iii) Suppose instead that each of two people has utility function $u(z)$ for money and current assets of £M. They have the opportunity to take part in a lottery L' in which they share any gains or losses when each independently plays the lottery L. Show that for each person L' is attractive provided

$$\lambda \leq \frac{2}{R} \log \frac{p}{1-p}.$$

5. (i) Observations $X_1, X_2 \ldots, X_n$ have a joint density function depending on an unknown scalar parameter θ with prior density $\pi(\theta)$. Explain what is meant by (a) the *risk function* of a decision rule d for estimating θ and (b) the *Bayes risk of d with respect to $\pi(\theta)$*, when the loss function is $L(\theta, d)$.

(ii) Two independent observations are made: X is Poisson with mean $\theta > 0$ and Y is Poisson with mean 2θ. The decision rule $d_{a,b}$ for estimating θ is given by

$$d_{a,b}(x, y) = aX + bY,$$

where a and b are constants.

(a) Find the risk function for $d_{a,b}$ for estimating θ under squared error loss and show that when θ has a prior exponential distribution with mean one, then the Bayes risk of $d_{a,b}$ is minimised with respect to a and b when $a = b = \frac{2}{7}$.

(b) If θ has the prior distribution in (ii)(a) above, find the Bayes Rule d_B for estimating θ and show that the ratio of the Bayes risk of $d_{\frac{2}{7},\frac{2}{7}}$ to that of d_B is 8:7.

Distribution	$f(x \mid \theta)$	$x \in \mathbb{X}$	$\theta \in \Theta$
Bernoulli(θ)	$\theta^x(1-\theta)^{1-x}$	$x = 0, 1$	$0 < \theta < 1$
(Discrete) Uniform(k)	$\frac{1}{k}$	$x = 1, 2, \dots, k$	
Binomial(n, θ)	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$	$x = 0, 1, \dots, n$	$0 < \theta < 1$
Poisson(λ)	$\frac{\lambda^x e^{-\lambda}}{x!}$	$x = 0, 1, 2, \dots$	$\lambda > 0$
Geometric(θ)	$(1-\theta)^{x-1}\theta$	$x = 1, 2, \dots$	$0 < \theta < 1$
NegativeBinomial(n, θ)	$\binom{x+n-1}{n-1}(1-\theta)^x\theta^n$	$x = 0, 1, 2, \dots$	$0 < \theta < 1,\ n = 1, 2, \dots$
Uniform(α, β)	$\frac{1}{\beta - \alpha}$	$\alpha < x < \beta$	$\alpha < \beta$
Exponential(λ)	$\lambda \exp(-\lambda x)$	$x > 0$	$\lambda > 0$
Gamma(ν, λ)	$\frac{1}{\Gamma(\nu)}\lambda(\lambda x)^{\nu-1}\exp(-\lambda x)$	$x > 0$	$\lambda > 0,\ \nu > 0$
Cauchy(α, β)	$\frac{1}{\pi\beta\left\{1 + \left(\frac{x-\alpha}{\beta}\right)^2\right\}}$	$-\infty < x < \infty$	$\beta > 0$
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$	$-\infty < x < \infty$	$\sigma^2 > 0$
Beta(α, β)	$\frac{1}{B(\alpha, \beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$0 < x < 1$	$\alpha > 0,\ \beta > 0$
Weibull(α, β)	$\beta\alpha x^{\alpha-1}\exp(-\beta x^\alpha)$	$x > 0$	$\alpha > 0,\ \beta > 0$
χ^2_k	$\frac{1}{2^{k/2}\Gamma(k/2)}x^{(k/2)-1}\exp(-\frac{1}{2}x)$	$x > 0$	$k = 1, 2, \dots$
t_m	$\frac{\Gamma((m+1)/2)}{\Gamma(m/2)\sqrt{\pi m}}\ (1 + \frac{x^2}{m})^{-(m+1)/2}$	$-\infty < x < \infty$	$m = 1, 2, \dots$
Pareto(θ)	$\frac{\theta}{x^{\theta+1}}$	$x > 1$	$\theta > 0$

Distribution	$E(X)$	$\mathrm{var}\,(X)$	$G_X(z)$ or $M_X(t)$
Bernoulli(θ)	θ	$\theta(1-\theta)$	$1-\theta+\theta z$
(Discrete) Uniform(k)	$(k+1)/2$	$(k^2-1)/12$	$z(1-z^k)/\{k(1-z)\}$
Binomial(n,θ)	$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta z)^n$
Poisson(λ)	λ	λ	$\exp\{-\lambda(1-z)\}$
Geometric(θ)	$\dfrac{1}{\theta}$	$\dfrac{1-\theta}{\theta^2}$	$\dfrac{\theta z}{1-z(1-\theta)}$
NegativeBinomial(n,θ)	$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta z}{1-z(1-\theta)}\right)^n$
Uniform(α,β)	$\frac{1}{2}(\alpha+\beta)$	$\frac{1}{12}(\beta-\alpha)^2$	$(e^{\beta t}-e^{\alpha t})/\{(\beta-\alpha)t\}$
Exponential(λ)	$1/\lambda$	$1/\lambda^2$	$\lambda/(\lambda-t)$
Gamma(ν,λ)	ν/λ	ν/λ^2	$\{\lambda/(\lambda-t)\}^\nu$
Cauchy	none	none	none
$N(\mu,\sigma^2)$	μ	σ^2	$\exp(\mu t+\frac{1}{2}\sigma^2 t^2)$
Beta(α,β)	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	${}_1F_1(\alpha;\beta;t)$
Weibull(α,β)	$\beta^{-1/\alpha}\Gamma\left(1+\dfrac{1}{\alpha}\right)$	$\beta^{-2/\alpha}\left\{\Gamma\left(1+\dfrac{2}{\alpha}\right) - \left[\Gamma\left(1+\dfrac{1}{\alpha}\right)\right]^2\right\}$	none
χ^2_k	k	$2k$	$(1-2t)^{-k/2}$
t_m	0	$\dfrac{m}{m-2}$	none
Pareto(θ)	$\dfrac{\theta}{\theta-1}$	$\dfrac{\theta}{(\theta-1)^2(\theta-2)}$	none