1. In a two-person zero-sum game the gain to player A when A plays pure strategy a and B plays pure strategy b is $g(a, b)$.

   (a) Explain what is meant by saying that (a, b) is a *pure strategy saddle-point*.

   (b) Two people together inherit an antique worth £400. Each independently makes a sealed bid. After the envelopes are opened the higher bidder inherits the antique and pays the other person an amount equal to the higher bid. If both make an equal bid then a fair coin is tossed to determine who shall inherit the antique and this person pays the other the amount of the (common) bid.

   If the only bids allowed are multiples of £100 (from £0 up to £400) find all pure strategy saddle-points (a, b).

   (c) For which of these pure strategy saddle-points (if any) are both a and b inadmissible?

   (d) Suppose now that any real numbered bid in the interval [0, 400] is allowed. By considering a particular pair of pure strategies of the form (a, b) where $b = a$, solve the game.

2. (a) Explain what is meant by an *equaliser strategy* in a two-person zero-sum game.

   (b) Show that if $\alpha^*$and $\beta^*$are equaliser strategies for the two players A and B respectively, then $\alpha^*$and β^* are maximin and minimax strategies for A and B respectively and the game has value $g(\alpha^*, \beta^*)$, the gain to A when A plays α^* and B plays β^*.

   (c) Army generals A and B command 3 and 2 regiments respectively and each wants to take over two forts, F_1 and F_2. Each general sends some regiments (possibly none) to fort F_1 and the rest to fort F_2. If a general sends more regiments to a particular fort than their opponent does, then they win a point for taking control of that fort (and their opponent loses a point). In addition the winning general gains (and their opponent loses) one point for each of the losing general's regiments at that fort. If they both send equal numbers of regiments to a fort there is no gain or loss to either side. For example if A and B send 1 and 2 regiments respectively to fort F_1 and 2 and 0 regiments respectively to fort F_2, then A loses 2 points at F_1 but gains 1 point at F_2, giving an overall pay-off to A of -1.

   Construct the pay-off matrix for this game.

   (d) Find equaliser strategies for the two generals and hence solve this game and show it has value $\frac{6}{5}$. You may find it helpful to exploit any symmetries of this game.

3. (a) In a two-person game explain what is meant by saying that a pair of randomised strategies is *in equilibrium*.

(b) Two drivers A and B drive their cars directly towards each other at high speed. Each driver can either keep on driving straight ahead (pure strategy D), daring the other to get out of the way, or can swerve (pure strategy S) to avoid a collision. If both drivers choose D they are both very badly injured and lose 20 points each. If both choose S they each lose one point for loss of face. If one chooses D and the other S, the driver choosing S loses one point but the one choosing D gains the admiration of their friends which is worth 2 points. There is no pre-play communication.

Show that this game has exactly two pairs of pure strategies in equilibrium

(c) Find a pair (α, β) of randomised strategies in equilibrium.

(d) Show that any point $(x, y) \in \mathbb{R}^2$ which is in the pay-off set takes the form

$$x = -22pq + 3p - 1, \quad y = -22pq + 3q - 1$$

for some $0 \leq p \leq 1, 0 \leq q \leq 1.$

(e) Find all values of $x \in \mathbb{R}$ such that the point (x, x) is in the pay-off set.

(f) Without further calculation sketch the pay-off set.

4. (a) Define a *utility function.*

(b) Explain why a person's utility function $u(z)$ for money, $z > 0$, is normally a bounded, concave function of z.

(c) An investor C wants to invest a sum of $£M$ $(M > 0)$ between two independent companies, A and B, for a year. For any sum $£a$ $(0 < a < M)$ invested in A there is a probability of 1/3 that this sum will all be lost at the end of the year and a probability of 2/3 that the sum will be doubled to $2a$. For any sum $£b$ $(0 < b < M)$ invested in B there is a probability of 2/3 that this sum will all be lost at the end of the year and a probability of 1/3 that the sum will be increased to $3b$.

C's utility for money is given by

$$u(z) = \begin{cases} z & 0 \leq z \leq 3M/2; \\ 3M/2 & z > 3M/2. \end{cases}$$

C invests $£a$ in A and $£b$ in B. Find the expected utility to C of this investment portfolio and show that the expected utility is maximised when $a = \frac{3}{4}M$ and $b = \frac{1}{4}M.$

5. (a) Each of two boxes, A and B, contains both red and black balls. One of the boxes contains equal numbers of red and black balls and the other contains a quarter red and three-quarters black balls, but you do not know which is which. If W denotes the box with equal numbers of red and black balls then $\mathsf{P}(W = A) = \theta$, $\mathsf{P}(W = B) = 1 - \theta$, where $0 < \theta < 1$, and θ is known.

You must choose one of the boxes and then select a ball (at random) from that box and, having noted its colour, you must then decide whether $W = A$ or $W = B$.

If R_A denotes the event of picking a red ball when box A is chosen, show that

$$\mathsf{P}(R_A) = \frac{1+\theta}{4} \quad \text{and} \quad \mathsf{P}(W = A \mid R_A) = \frac{2\theta}{1+\theta}.$$

(b) Construct a decision tree for this problem, showing all relevant probabilities.

(c) Show that if $1/2 < \theta < 2/3$ then in order to maximise your probability of a correct decision, you should select a ball from box B. Describe the decision you would then make having selected a ball from this box.