UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY– JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M3S11/M4S11 GAMES RISKS AND DECISIONS

DATE: Wednesday, 4th June 2003 TIME: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1. *a)* For a two person zero-sum game, explain what is meant by an *inadmissible strategy* and by an *equaliser strategy*.

Explain briefly without proof why a pair (α, β) of equaliser strategies is useful in solving finite two person zero-sum games.

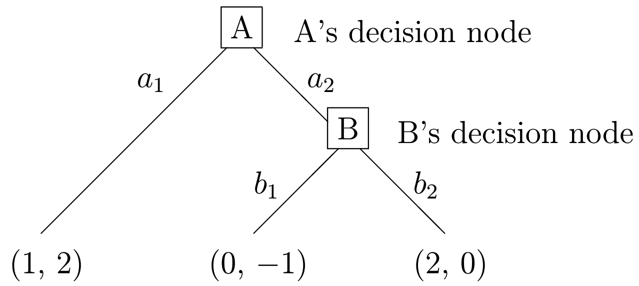
b) Solve the following two person zero-sum games having payoff matrices G_1 and G_2. Give your reasoning. There is no need to state or prove any theorems that you use.

$$G_1 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & -5 & 5 & -2 \\ -5 & 5 & -2 & 1 \\ 5 & -2 & 1 & -5 \\ -2 & 1 & -5 & 5 \end{pmatrix} \end{array} \qquad G_2 = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{pmatrix} -2 & -2 & 0 & -1 & 1 \\ 2 & 1 & -2 & -3 & -2 \\ -1 & 1 & -2 & -4 & -1 \end{pmatrix} \end{array}$$

2. *a)* For a non-zero sum two person game between players A and B explain what is meant by the players having an *equilibrium pair of strategies.*

What consequence does this have for the players?

b) Consider the game in extensive form shown below, where A chooses between strategies a_1 and a_2, and where, if A has chosen strategy a_2, B chooses between strategies b_1 and b_2. The payoffs $(g_A\,,\ g_B)$ to A and B are shown at the tips of the decision tree. Solve the game.



c) Consider the non-zero sum two person game in strategic form having payoff matrix

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \quad b_1 \qquad\quad b_2 \\ \begin{pmatrix} (1,2) & (1,2) \\ (0,-1) & (2,0) \end{pmatrix} \end{array}$$

i) Obtain the solution to this game.

ii) Find the maximin solutions $(\alpha^*,\ \beta^*)$ for A and B.

iii) Show that $(\alpha^*,\ \beta^*)$ is not an equilibrium pair of strategies.

iv) Find two pairs of pure strategies that are equilibrium pairs, and a third equilibrium pair involving a randomised strategy.

v) Is the game Nash solvable? Give a reason for your answer.

3. *a)* What is meant by saying that a consistent preference ordering has an agreeing utility function?

b) Consider the following properties of a consistent preference ordering.

(I) If $p_1 \prec p_2$ and $0 \le \nu < \mu < 1$ then

$$\mu p_1 + (1-\mu)p_2 \prec \nu p_1 + (1-\nu)p_2.$$

(II) If $p_1 \prec p_2 \prec p_3$, $\exists$ a unique η $(0 < \eta < 1)$ for which

$$p_2 \approx \eta p_1 + (1-\eta)p_3.$$

Show that these statements make sense in terms of agreeing utilities.

c) A gamble $\{y;\ y_1,\ y_2;\ p\}$ takes the gambler's wealth y to y_1 with probability p, or to y_2 with probability $1-p$.

A gambler, who is free to choose x $(0 \le x \le y)$, is to take the gamble $\{y;\ y+x,\ y-x;\ p\}$, where the choice $x = 0$ is permitted.

The gambler has the utility function

$$U(r) = \begin{cases} r & (0 \le r \le 1); \\ r^2 & (1 \le r < \infty). \end{cases}$$

Show how the gambler should choose x in the two cases $y = 0.5$ and $y = 2$, considering in each case every value of p $(0 < p < 1)$.

Find the maximum expected utility in each case.

4. *a)* Show that the Bayes decision in estimating θ by $\widehat{\theta}(x)$ under squared error loss is obtained when $\widehat{\theta}(x)$ is the expectation of the posterior distribution given x.

b) Let the distribution of X given $\Theta = \theta$ be Binomial$(3, \theta)$, and let the prior distribution of Θ be Beta$(3, 1)$.

i) Suppose that we have a single observation x of X.
Describe the posterior distribution of Θ given $X = x$.
Write down the expectation and variance of this posterior distribution.
Determine the mode of the posterior probability density function.
Write down the Bayes estimate of θ under squared error loss.

ii) Consider the classical decision rule $d_c(x) = cx$ under squared error loss for this problem.
Determine the classical risk function $R(\theta,\, d_c(x))$.
Determine the value of c that minimises the classical risk function and find the minimum risk.

[*Binomial* (n, θ) has probability mass function

$$p_x = \binom{n}{x} \theta^x \,(1-\theta)^{n-x} \quad (x = 0, 1, \ldots, n)$$

for $0 < \theta < 1$, with $EX = n\theta$ and $\operatorname{var} X = n\theta(1-\theta)$.
Beta (α, β) has probability density function

$$f(x) = \frac{1}{B(\alpha, \beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1} \quad (0 < x < 1)$$

for $\alpha > 0$ and $\beta > 0$, with

$$EX = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \operatorname{var} X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} . \qquad]$$

5. Sales of a new product can be in one of three states

- θ_0 : failure, • θ_1 : moderate success, • θ_2 : high success.

The actions that are available to the marketing department are advertising using

- a_0 : bill boards, • a_1 : newspapers, • a_2 : television.

The loss table is judged to be (in millions of pounds)

$$\Big(L(\theta_i\,,\,a_j) \Big) = \begin{array}{c} \\ \theta_0 \\ \theta_1 \\ \theta_2 \end{array} \begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{pmatrix} 5 & 10 & 10 \\ -10 & -5 & 5 \\ -20 & -15 & -25 \end{pmatrix} \end{array}$$

a) Assuming a prior distribution $(\pi_0\,,\ 1-\pi_0-\pi_2\,,\ \pi_2)$ for $(\theta_0\,,\theta_1\,,\theta_2)$, determine what action should be taken for each possible $(\pi_0\,,\pi_2)$.

Sketch the regions in the $(\pi_0\,,\pi_2)$ plane showing their corresponding appropriate actions.

b) Suppose that for zero cost a consultancy reports C, a prediction of θ_2, high success.

The probabilities $P(C \mid \theta_i)$ are known and given by

$$P(C \mid \theta_0) = 0.2\,, \quad P(C \mid \theta_1) = 0.6\,, \quad P(C \mid \theta_2) = 0.8\,.$$

Determine the posterior probabilities $(\pi_0',\ 1-\pi_0'-\pi_2',\ \pi_2')$ in terms of π_0 and π_2, where $\pi_i' = P(\theta_i \mid C)$.

Hence, given C, determine the action to be taken for each possible $(\pi_0\,,\pi_2)$.

Sketch the regions in the $(\pi_0\,,\pi_2)$ plane showing their corresponding actions.