Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P9/M4P9

Linear Algebra and Matrices

Date: Wednesday, 10th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Note: *In this paper, R denotes a commutative ring, K a field, and $K[X]$ is the polynomial ring over K.*

1. Let A and B be $r \times r$ matrices over K. Say what is meant by 'A and B are *conjugate*'.

   Let $M = (K^r, A)$ and $N = (K^r, B)$ be the corresponding $K[X]$-modules. Show that the $K[X]$-module homomorphisms $\theta : M \to N$ are given by the matrices T such that $TA = BT$. Deduce that $M \cong N$ as $K[X]$-modules if and only if A, B are conjugate.

   Show also that, if A and B are conjugate, then $c_A(X) = c_B(X)$.

   Now take $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and let M, N, P respectively be the corresponding $\mathbb{R}[X]$-modules. Determine whether or not any of M, N, P are isomorphic to one another as $\mathbb{R}[X]$-modules.

   Does your answer change if we replace $\mathbb{R}$ by $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, the field of two elements? Give your reasoning.

2. Define the following terms.

   (1) A *cyclic* R-module.

   (2) The *annihilator* $\operatorname{Ann}(M)$ of an R-module M.

   Show that M is cyclic if and only if there is a surjective R-module homomorphism $\pi : R \to M$. Show also that the kernel of π is $\operatorname{Ann}(M)$, and that $R/\operatorname{Ann}(M) \cong M$ as R-modules. If you use an Isomorphism Theorem, then you must prove that theorem.

   Show also that if N is another cyclic R-module, then $M \cong N$ as R-modules if and only if $\operatorname{Ann}(M) = \operatorname{Ann}(N)$.

   Now take $R = K[X]$, let

   $$g(X) = X^r - g_{r-1}X^{r-1} - \cdots - g_1 X - g_0 \in K[X], \quad r \geq 1$$

   and put $M = K[X]/K[X]g(X)$.

   Show that M is a vector space over K, and find a basis of M so that the action of X on M is represented by the companion matrix $C(g)$ of $g(X)$.

3. Let M be an R-module. State what is meant by 'M is the direct sum of its submodules L, N'. Show that the following are equivalent.

(1) $M = L \oplus N$;

(2) Given $m \in M$, there are unique elements $\ell \in L$, $n \in N$ with $m = \ell + n$.

Let A be an $r \times r$ matrix over a field K and let $M = (K^r, A)$ be the $K[X]$-module given by A. Show that $M = L \oplus N$ with $\dim_K(L) = t$ and $\dim_K(N) = s$ if and only if there is an invertible matrix P with $P^{-1}AP = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ for some matrices B and C, of sizes $s \times s$ and $t \times t$ respectively.

Suppose that M is indecomposable, that is, $M = L \oplus N$ only if $L = 0$ or $N = 0$. Show that $m_A(X) = p(X)^k$ for some irreducible polynomial $p(X)$. (You may use without proof a result that leads to a direct sum decomposition, but you must state it clearly.)

Is the converse true?

Take $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, regarded as a matrix over the complex numbers $\mathbb{C}$.

Determine whether or not M is indecomposable.

4. Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, let $M = (\mathbb{C}^5, A)$ and let $N = (\mathbb{R}^5, A)$.

Find the primary decomposition of M as a $\mathbb{C}[X]$-module, giving a $\mathbb{C}$-basis for each component.

Hence find the primary decomposition of N as an $\mathbb{R}[X]$-module, giving an $\mathbb{R}$-basis for each component.

Find also the Rational Canonical Forms for A over $\mathbb{C}$ and over $\mathbb{R}$. (You should not need to make any hard calculations for this last part.)

5. Let M be a finitely generated module over $K[X]$, and suppose that M is $p(X)$-primary for some irreducible polynomial over K.

State the Elementary Divisor Theorem for M.

Suppose further that $p(X) = X - \lambda$ and that $M = K[X]/(X-\lambda)^t K[X]$ for some $t \geq 1$. Construct the K-basis $v_1, \ldots, v_t$ of M such that the action of X on M is described by a Jordan Block matrix.

Describe the Jordan Normal Form of a complex matrix A.

Given that a complex matrix A has a Jordan Normal Form, show that A can be written as a sum $A = D + N$, where D is diagonalizable, N is nilpotent, and $DN = ND$.