1. Let A be an $r \times r$ matrix over a field F. Show how to define the scalar multiplication between a polynomial $f(X) \in F[X]$ and a vector $v \in F^r$ that gives rise to the $F[X]$-module $M = (F^r, A)$ with "X acting as A". (You are not expected to verify that M is a module.)

Let B be an $s \times s$ matrix over F and let $N = (F^s, B)$. Show that there is a bijective correspondence between

(1) $F[X]$-module homomorphisms $\theta : M \to N$

and

(2) $s \times r$ matrices T over F with $TA = BT$.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$. Find all $F[X]$-module homomorphisms from M to N.

2. Let F be a field and let M be an $F[X]$-module. Define an $F[X]$-*submodule* of M.

Take $M = (F^r, A)$ for a matrix A over F. Show that the $F[X]$-submodules of M are in bijective correspondence with the A-invariant subspaces of F^r.

Suppose that L is a submodule of M and that $L \neq 0, M$. Show that there is an invertible matrix P with $P^{-1}AP = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$.

Prove further that $M = L \oplus N$ for some submodule N if and only if we can take $D = 0$.

3. Define

(a) the *minimal polynomial* $m_A(X)$ of an $r \times r$ matrix over a field F;

(b) the *annihilator* $\mathrm{Ann}(M)$ of an $F[X]$-module M.

Establish the relationship between $m_A(X)$ and $\mathrm{Ann}(M)$ when $M = (F^r, A)$.

<u>State</u> the Cayley-Hamilton Theorem, and deduce that $m_A(X)$ divides the characteristic polynomial $c_A(X)$.

<u>State</u> a further relationship between $m_A(X)$ and $c_A(X)$, and deduce that an irreducible polynomial $p(X)$ divides $m_A(X)$ if and only if it divides $c_A(X)$

For each $i = 1, \ldots, r$, give, with proof, an example of an $r \times r$ matrix with minimal polynomial of degree i.

4. Let M be an $F[X]$-module, F a field, and suppose that $b(X), c(X)$ are coprime polynomials with $b(X)c(X)M = 0$. Show that M has a direct sum decomposition $M = b(X)M \oplus c(X)M$.

Let $p(X)$ be an irreducible polynomial. Define a $p(X)$-*primary* module.

Given that $\mathrm{Ann}(M) \neq 0$, prove that $M = M_1 \oplus \cdots \oplus M_k$ with each M_i $p_i(X)$-primary for some $p_i(X)$.

Let $A = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Find the primary decomposition of $(\mathbb{C}^4, A)$. (You are *not* expected to find F-bases of the components.)

5. Let $J = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$ be a $t \times t$ Jordan block matrix. Show that

$$m_J(X) = c_J(X) = (X - \lambda)^t,$$

Determine all the possible Jordan Normal Forms of an $r \times r$ complex matrix that satisfies an equation $A^n = I_r$ for some $n > 1$.