Question 1. Let X and Y be two normed linear spaces and let $T : X \to Y$ be a bounded linear operator from X to Y.

a. Define the norm of the operator T.

b. Find the norm of the operator $T : L^p(0,2) \to L^p(0,2)$, $1 \le p < \infty$, where

$$Tf(t) = t^2 f(t), \qquad t \in (0,2).$$

Question 2. Let X and Y be metric linear vector spaces and let $T : X \to Y$ be a linear operator from X to Y.

a. Define the graph of the operator T.

b. Give definition of a closed operator.

c. Show that the operator T in $L^2(0,1)$ with $\mathcal{D}(T) = C[0,1]$, such that

$$Tf = f(0),$$

is not closable.

Question 3.

a. What is the Schwartz inequality in a Hilbert space? Prove it.

b. What is the Schwartz inequality for the Hilbert space $L^2(\mathbb{R})$.

c. What is Bessel's inequality? Prove it.

Question 4. Let $H^1(0,2\pi) \subset L^2(0,2\pi)$ be a Hilbert space with the scalar product given by

$$(f,g) = \int_0^{2\pi} \left(f'(x)\overline{g'(x)} + f(x)\overline{g(x)} \right) dx$$

and let

$$e_n(x) = a_n e^{inx}.$$

a. Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthogonal system in $H^1(0,2\pi)$. Find the coefficients a_n, such that it is an orthonormal system.

b. Find a function $f \in H^1(0,2\pi)$ such that f is orthogonal to all e_n, $n \in \mathbb{Z}$, in $H^1(0,2\pi)$.

Question 5.

a. Show that $T : L^2(0, \infty) \to L^2(0, \infty)$ defined by

$$Tf(x) = \int_0^\infty \frac{1}{x+y} f(y)\, dy$$

is bounded.

b. Show that T is not compact.