Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P7/M4P7

Functional Analysis

Date: Thursday, 25th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Give the definitions of a linear metric space, a normed space and a Banach space.

   (b) Give an example (no proof is required)

      (i) of a linear metric space which is not a normed space;

      (ii) of a normed space which is not a Banach space.

   (c) Define what it means to say that a metric linear space is

      (i) complete;

      (ii) separable.

   (d) Prove or disprove that the space $(l_p, ||\cdot||_p)$ is complete for $p \in [1, \infty)$.

2. (a) Give a definition of the Hamel basis and Schauder basis and explain differences between them, if any.

   (b) Which of the following spaces does not have a Hamel basis and which does not have a Schauder basis? Justify briefly your answer.

      (i) $(l_p, ||\cdot||_p)$, $p \in [2, 3)$;

      (ii) $(l_\infty, ||\cdot||_\infty)$;

      (iii) $(c_0, ||\cdot||_\infty)$;

      (iv) $(c, ||\cdot||_\infty)$;

      (v) a separable Hilbert space.

   (c) Prove that every continuous function on an interval $[a, b]$ can be approximated in a uniform norm by a sequence of polygonal functions.

3. *In this question any general result may be used without justification.*

   (a) Prove that for any nonzero vector x in a Banach space $(X, ||\cdot||)$, over the field of real numbers, there exists a bounded linear functional F of norm one such that $F(x) = ||x||$. State clearly any result used.

   (b) Prove that the functional $f : c \to \mathbb{R}$ given by $f(x) \equiv \lim_{n\to\infty} x_n$ can be extended to the entire space l_∞. Thereby show that to any bounded sequence one can associate a notion of a limit.

   Hint: Consider $p(x) \equiv \limsup_{n\to\infty} |x_n|$ *which is well defined on* l_∞.

   (c) Prove or disprove the following statement.

   If a sequence Ψ_n, $n \in \mathbb{N}$, in the normed space $(\mathbb{L}_2, ||\cdot||_2)$, is convergent weakly to an element $\Psi \in \mathbb{L}_2$, then the sequence of norms $(||\Psi_n||)_{n\in\mathbb{N}}$ is bounded.

   Hint: Consider a suitable sequence of operators on the dual space.

4. (a) Give the definition of a closed operator and that of a compact operator.

   (b) (i) With justification, give an example of a bounded operator which is not closed.

   (ii) With justification, give an example of unbounded closed operator.

   (c) Prove that the following operator is compact $T : l_p \to l_p$, for some $p \in [1, \infty)$,

$$T(\mathbf{x}) \equiv \sum_{n=1,\dots,N} \mathbf{y}_n(\mathbf{x})\mathbf{e}_n$$

   where $N \in \mathbb{N}$, and, for $n = 1, \dots, N$, $\mathbf{y}_n \in (l_p)^*$ are given elements of the dual space and $\mathbf{e}_n$ are elements of the canonical basis in l_p.

5. Which of the following statements are true and which are not? No reasoning need be given: just answer T or F for each.

(5.1) Some linear spaces have no Hamel basis.

(5.2) The linear space $\mathbb{C}^n$ over the field $\mathbb{Q}$ is finite dimensional.

(5.3) In each metric linear space, multiplication by a scalar is not continuous.

(5.4) In every normed space the addition of vectors is not continuous.

(5.5) All norms on a finite dimensional space are equivalent.

(5.6) If a unit ball in a metric linear space is compact, then the space is finite dimensional.

(5.7) Every linear operator from a finite dimensional normed space into another normed space is bounded.

(5.8) Weierstrass theorem implies that the set of monomials form a Schauder basis in $C([0,1])$.

(5.9) There exist two different Banach spaces whose topological dual spaces are isometrically isomorphic.

(5.10) The topological dual space of c_0 is isometrically isomorphic to l_1.

(5.11) The space l_2 is isometrically isomorphic to any infinite dimensional separable Hilbert space.

(5.12) The sequence given by the canonical Schauder basis in l_p, $p \in (1, \infty)$, converges in the weak topology.

(5.13) A nonempty complete metric space is of second category.

(5.14) Banach - Steinhaus theorem is not true in finite dimensional normed spaces.

(5.15) There exists a continuous real valued periodic function whose Fourier series diverges at any given point.

(5.16) An unbounded linear operator cannot be closed.

(5.17) Not every closed operator is bounded.

(5.18) The graph of closed operator can be open in the product topology.

(5.19) If $(X, ||\cdot||)$, $X \neq \emptyset$, is a Banach space, then X is of first category.

(5.20) The inverse of the bijective operator acting between two Banach spaces may be unbounded.