UNIVERSITY OF LONDON

Course: M3P7/M4P7
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M3P7/M4P7 FUNCTIONAL ANALYSIS

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

1. **(a)** Define a metric linear space.

(b) Which of the following spaces satisfy conditions of the metric linear space.

(i)

$$s \equiv \{x \equiv (x_j \in \mathbb{R})_{j\in\mathbb{N}} : \sum_{k\in\mathbb{N}} |x_k|^k < \infty\}$$

with coordinatewise addition of the vectors and multiplication by a scalar and a metric

$$\rho(x,y) \equiv \sum_k 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

(ii) $\mathbb{M}_2$ complex 2×2 matrices with a metric given as follows

$$\rho(m_1, m_2) \equiv \left(Tr((m_1 - m_2)^*(m_1 - m_2))\right)^{\frac{1}{2}}$$

where $Tr(A)$ denotes the trace of a matrix A and m^* denotes the adjoint matrix of m.

2. **(a)** State the Banach contraction mapping principle explaining carefully all necessary notions.

(b) Prove or disprove that the following equation has a unique solution in any interval $[0,T]$ for each $T \in \mathbb{R}^+$.

$$x(t) = \cos(\pi t) + 10^{23} \int_0^t ds \sin\left((t-s)|x(s)|\right)$$

3. (a) State the Banach - Steinhaus theorem.

(b) Give a definition of the weak convergence in a Banach space and explain a difference between weak and strong convergence, if any.

(c) Prove or disprove the following statement: *If a sequence x_n in a normed space $(X, ||\cdot||)$ is convergent weakly to an element $x \in X$, then the sequence of norms $(||x_n||)_{n\in\mathbb{N}}$ is bounded.*

Hint: Consider a suitable sequence of operators on the Banach space $(X^*, ||\cdot||_{X^*})$.

4. (a) Give a definition of a closed operator.

(b) Consider the following operator $A : \mathcal{D} \to \mathcal{C}([-\pi, \pi])$

$$Af(x) \equiv (\frac{d}{dx} + x)f(x)$$

on the domain $\mathcal{D}$ once continuously differentiable functions vanishing at the end points of the interval $[-\pi, \pi]$. Assuming the supremum norm in the space, prove or disprove that this operator is

(i) closed;

(ii) unbounded.

(c) Does any of this property change if the space of continuous functions will be replaced by the square integrable functions on $[-\pi, \pi]$.

5. Which of the following statements are true and which are not ? No reasoning need be given: just answer T or F in the box next to each question. The marking will be as follows (subject to a minimum score of zero) : each correct answer scores +1, each wrong answer -1 and zero for no answer.

(5.1) ☐ Each linear space has a Hamel basis.

(5.2) ☐ The linear space $\{a + b\sqrt{-3} : a, b \in \mathbb{Q}\}$ is one dimensional.

(5.3) ☐ Each metric linear space is a normed space.

(5.4) ☐ In every normed space the addition of vectors is continuous, but multiplication by a scalar may be not continuous.

(5.5) ☐ All norms on a finite dimensional space are equivalent.

(5.6) ☐ If a unit ball in a metric linear space is compact, then the space is finite dimensional.

(5.7) ☐ Every linear operator from a finite dimensional normed space into itself is bounded.

(5.8) ☐ There exists a linear operator between two Banach spaces which is continuous, but not bounded.

(5.9) ☐ There exists two different Banach spaces which topological dual spaces are isometrically isomorphic.

(5.10) ☐ The topological dual space of l_∞ is isometrically isomorphic to l_1.

(5.11) ☐ The topological dual space of l_2 is isometrically isomorphic to any infinite dimensional separable Hilbert space.

(5.12) ☐ A sequence given by the canonical Schauder basis in l_p, $p \in (1, \infty)$, converges in the weak topology.

(5.13) ☐ A nonempty complete metric space is of second category.

(5.14) ☐ Banach - Steinhaus theorem is not true in finite dimensional normed spaces.

(5.15) ☐ There exists a continuous real valued periodic function whose Fourier series diverges at any given point.

(5.16) ☐ An unbounded linear operator cannot be closed.

(5.17) ☐ Each bounded operator is closed.

(5.18) ☐ The graph of closed operator can be open in the product topology.

(5.19) ☐ If $(X, ||\cdot||)$, $X \neq \emptyset$, is a Banach space, then X is of first category.

(5.20) ☐ Inverse of the bijective operator acting between two Banach spaces may be unbounded.