Q1. (i) Give a definition of a probability space explaining carefully all notions involved.

(ii) Give a definition of convergence in probability and almost everywhere, and explain relationship between them.

(iii) Explain giving reasons which of the following sequences converges in probability and which converges almost everywhere on the probability space $([0,1],\Sigma,\lambda)$ with λ denoting the Lebesgue measure.

(a) Let $\mathbb{Q} \equiv \{q_n \in [0,1]\}_{n\in\mathbb{N}}$ be enumeration of rational numbers in the unit interval. Let

$$f_n(x) \equiv \begin{cases} 1 & if\ x = q_n \\ 0 & otherwise \end{cases}$$

(b)

$$f_n(x) = 2^n \chi([\sin(n\frac{\pi}{2}) - 2^{-n}, \sin(n\frac{\pi}{2}) + 2^{-n}])$$

where $\chi([a,b])$ denotes the characteristic function of the interval $[a,b]$.

2. (i) Suppose that for a random variable f on a probability space (Ω,Σ,μ) and for some $\varepsilon \in (0,\infty)$, $e^{\pm\varepsilon f}$ is integrable. Show that for $L > 0$

$$\mu(\{|f| > L\}) \leq e^{-\varepsilon L} 2\mu(\cosh(\varepsilon f))$$

(ii) Let $\mu \equiv \nu_0^{\otimes\mathbb{N}}$ be the infinite product of symmetric Bernoulli measures ν_0, on the space (Ω,Σ) with $\Omega \equiv \{-1,+1\}^{\mathbb{N}}$ and Σ denoting the Borel σ-algebra. Assuming Poincare inequality for the infinite product of symmetric Bernoulli measures prove the WLLN for the following sequence of dependent random variables: $X_n \equiv \pi_n \cdot \pi_{n+1} \cdot \pi_{n+2}$, $n \in \mathbb{N}$, where $\Omega \ni \omega \equiv (\omega_j \in \{-1,+1\})_{j\in\mathbb{N}} \longmapsto \pi_n(\omega) \equiv \omega_n$.

3. (i) State the Strong Law of Large Numbers for i.i.d. random variables with finite second moment.

(ii) Prove the Strong Law of Large Numbers for i.i.d. random variables which have finite 5-th moment.

4. (i) Give a definition of the distribution function and a characteristic function of a random variable X on a probability space (Ω, Σ, μ)

(ii) Prove or disprove that the characteristic function is positive definite, continuous and equal to one at zero.

(iii) Prove or disprove that a linear combination of two independent Gaussian random variables is also a Gaussian random variable.

5. (i) Find the characteristic function $\xi_n(t)$ of

$$S_n = \sum_{j=1}^{n} (X_j - < X_j >)$$

where X_j are independent Bernoulli random variables and $< X_j >$ denotes the mean value of X_j.

(ii) Using the result in Part (i), prove or disprove that

$$\xi_n(\frac{t}{\sqrt{n}}) \to e^{-\frac{1}{2}\sigma^2 t^2}$$

for some $\sigma^2 \in (0, \infty)$.

(iii) Using this and assuming the Poincare inequality for the product of Bernoulli measures prove the Poincare inequality for the Gaussian measure on real line.