Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P6/M4P6

Probability Theory

Date: Wednesday, 10th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) State the definition of the *variance* of an integrable random variable.

   (ii) Determine the variance of a random variable which is uniformly distributed on the interval $(1, 2)$.

   (iii) Let $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(0, 1)$ and $P =$ Lebesgue measure on $(0, 1)$. Show that

   (a) $\infty = \sup\{\text{var}(X) | X : \Omega \to \mathbb{R} \text{ is a random variable with } E(X) = 1\}$.

   (b) $\infty \neq \sup\{\text{var}(X) | X : \Omega \to \mathbb{R} \text{ is a random variable with } E(X^2) = 1\}$.

2. (i) State the definition of what it means for a family of random variables $(X_i)_{i \in I}$ to be *independent*, where I is an arbitrary index set.

   (ii) Let X and Y be two random variables with joint distribution described by the following table:

$Y \backslash X$	-1	1	0
0	0	0	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{4}$	0

   Show that X and Y are uncorrelated. Are X and Y independent? Prove your answer.

3. Let $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(0, 1)$ and $P =$ Lebesgue measure on $(0, 1)$. Let $s \geq 0$ and $X_n(\omega) = n^s \, 1_{(0, \frac{1}{n})}(\omega)$, $n \geq 1$.

   (i) Show that the sequence (X_n) converges in probability.

   (ii) For which $s \in [0, \infty)$ does (X_n) also converge in L^1? Prove your answer.

   (iii) Assume $s = 0$ and define $Y_n(\omega) = X_1(\omega) + \ldots + X_n(\omega)$. Show that Y_n converges almost surely, but *not* in L^1.

   (iv) Now let Z, Z_n be arbitrary real-valued random variables and $p \geq 1$. Show that if (Z_n) converges to Z in L^p, then it converges to Z in probability as well.

4. Let X, X_n be real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$.

(i) Show that the event $\{\lim_n X_n = X\}$ belongs to $\mathcal{F}$.

(ii) Let (X_n) be *independent* and $A = (a, b)$ an arbitrary interval. Show with the help of both Borel-Cantelli Lemmas that either $P(X_n \in A \text{ i.o.}) = 0$ or $P(X_n \in A \text{ i.o.}) = 1$.

(iii) Let again (X_n) be independent and $A = (a, b)$ an arbitrary interval. State whether the following implications are correct or not. Prove your answers.

(a) $P(X_n \in A \text{ i.o.}) = 1 \Longrightarrow P(X_n \in A^c \text{ i.o.}) = 0.$

(b) $P(X_n \in A \text{ i.o.}) = 0 \Longrightarrow P(X_n \in A^c \text{ i.o.}) = 1.$

5. (i) Let X be a random variable such that $P(X = 1) = \frac{1}{2} = P(X = -1)$. Compute the characteristic function of X.

(ii) State the definition of a martingale.

(iii) Let $Y, X_1, X_2, X_3, \ldots$ be a collection of independent and bounded random variables such that $E(X_n) = 0$ for all $n \geq 1$. Let $\mathcal{F}_n = \sigma(Y, X_1, \ldots, X_n)$ be the σ-algebra generated by $Y, X_1, X_2, \ldots, X_n$. Show that $Z_n = Y(X_1 + \cdots + X_n)$ is a martingale with respect to $(\mathcal{F}_n)$, for $n \geq 1$.