M3/4P6 Exam 2005

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

   (a) State the definition of a probability measure P.

   (b) Let $\{E_n\}$ be a sequence of events in $\mathcal{F}$. Prove that

   $$P(\bigcup_n E_n) \leq \sum_n P(E_n).$$

   (c) Let the distance of two events A and B be defined by $d(A,B) = P(A\Delta B)$, where $A\Delta B$ is the symmetric difference of A and B. Prove that d satisfies the triangle inequality. Furthermore, prove that if A and B are independent, then

   $$d(A,B) \geq (P(B) - P(A))^2.$$

2. Let $X_1, \ldots, X_n$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

   (a) State the definition of $X_1, \ldots, X_n$ being independent.

   (b) Let X_1, X_2, X_3 be independent random variables with $P(X_i = 0) = P(X_i = 1) = 1/2$. Let

   $$A_1 = \{X_2 = X_3\}, \quad A_2 = \{X_3 = X_1\}, \quad A_3 = \{X_1 = X_2\}.$$

   Show that these events are pairwise independent but not independent.

   (c) Let X_1 and X_2 be independent and have uniform distributions on $(0,1)$. Let

   $$W_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2), \quad W_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2).$$

   Show that W_1 and W_2 are independent and have standard normal distributions.

3. Let $\{X_n\}$ be a sequence of random variables and X a random variable.

   (a) State the following definitions: convergence in probability ($X_n \xrightarrow{P} X$), convergence almost surely ($X_n \xrightarrow{\text{a.s.}} X$), convergence in L^p norm ($X_n \xrightarrow{L^p} X$), and convergence in distribution ($X_n \xrightarrow{D} X$).

   (b) Draw a diagram to illustrate the relationship between these modes of convergence.

   (c) Prove that if $X_n \xrightarrow{P} X$ then there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$ as $k \to \infty$.

4. (a) Let $\{X_n\}$ be a sequence of independent random variables with $EX_i = \mu$ and $\text{var}(X_i) \leq C < \infty$. Prove that

   $$\frac{S_n}{n} \xrightarrow{L^2} \mu$$

   where $S_n = X_1 + \cdots + X_n$.

(b) Let f be a continuous function on $[0,1]$ and let

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, be the Bernstein polynomial of degree n associated with f. Prove that as $n \to \infty$,

$$\max_{x \in [0,1]} |f_n(x) - f(x)| \to 0.$$

5. Let $\{X_n\}$ be a sequence of random variables and X a random variable.

(a) Prove that if $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{D} X$.

(b) Prove that if $X_n \xrightarrow{D} c$ and c is a constant than $X_n \xrightarrow{P} X$.

(c) Let $\{X_n\}$ be independent and identically distributed with $E(X_1) = m < \infty$. Prove that

$$\frac{S_n}{n} \xrightarrow{P} m$$

where $S_n = X_1 + \cdots + X_n$.