1. A compact connected surface S has a subdivision with V vertices, E edges and F faces.

   (i) Define the Euler characteristic $\chi(S)$ of S.

   (ii) Suppose that each vertex lies on at most 4 edges and that no edge starts and ends at the same vertex. Let s denote the number of vertices which lie on at most 3 edges. Show that $s \leq 4V - 2E$.

   (iii) Suppose moreover that each face has four sides. Show now that $s \leq 4\chi(S)$. Hence deduce that no such subdivision is possible for a sphere with g handles when $g \geq 2$.

   (iv) Give an example to show that there is such a subdivision of the torus.

2. Let $r\colon V \to \mathbb{R}^3$ parametrise a smooth patch of surface with first fundamental form $E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$.

   (i) Prove that r is area-preserving if and only if $EG - F^2 = 1$.

   (ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth and strictly positive. Let $g\colon \mathbb{R} \to \mathbb{R}$ be smooth and strictly increasing. The map $r(u,v) = (g(u), f(g(u))\cos v, f(g(u))\sin v)$ parametrises the surface of revolution corresponding to f.

      (a) Prove that r is area preserving if and only if

      $$g'^2\left(1 + (f' \circ g)^2\right)(f \circ g)^2 = 1$$

      (b) Let $h\colon \mathbb{R} \to \mathbb{R}$ be smooth function satisfying

      $$h' = (1 + f'^2)^{1/2} f.$$

      Explain briefly why h is invertible.

      (c) Prove that taking g to be the inverse to h makes the parametrisation r area preserving

3. Let $r\colon V \to \mathbb{R}^3$ parametrise a smooth patch of surface S, with positive unit normal $n\colon V \to S^2$. Write the first and second fundamental forms of S as $E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$ and $L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$ respectively.

   (i) State a formula for the Gaussian curvature of S in terms of the coefficients of the two fundamental forms.

   (ii) Let $f\colon V \to \mathbb{R}$ be a smooth function with $f(u,v) \geq 0$ for all $(u,v) \in V$ and $f(u_0,v_0) = 0$. Consider the map $\widetilde{r}\colon V \to \mathbb{R}^3$ given by $\widetilde{r} = r + fn$. You may assume that $\widetilde{r}$ parametrises a smooth patch of surface $\widetilde{S}$.

      (a) Prove that at (u_0,v_0) the positive unit normals to S and $\widetilde{S}$ agree.

      (b) Let $K, \widetilde{K}$ denote the Gaussian curvatures of $S, \widetilde{S}$ respectively. Prove that

      $$\widetilde{K}(u_0,v_0) = K(u_0,v_0) + \frac{f_{uu}f_{vv} - f_{uv}^2}{EG - F^2} + \frac{Lf_{vv} + Nf_{uu} - 2Mf_{uv}}{EG - F^2}$$

      where all terms on the right hand side are evaluated at (u_0,v_0).

4. (i) (a) State a theorem concerning the existence and uniqueness of geodesics.

(b) Describe, with justification, all the geodesics on the unit sphere S^2.

(ii) You may *not* quote any form of the Gauss–Bonnet theorem in answering this question. You may assume that S^2 has area 4π.

(a) Let $a, b \in S^2$ be diametrically opposite points. Let γ_1 and γ_2 be geodesics of length π, each starting at a and ending at b and making an angle α at a. Prove that the area of the wedge-shaped region enclosed by γ_1 and γ_2 is 2α.
[Hint: What proportion of the sphere does the wedge cover?]

(b) Let $p, q, r \in S^2$ be distinct points. Let T be the triangular region whose boundary is made up by the geodesics joining p, q, r in pairs. Let the interior angles of T be $\alpha_1, \alpha_2, \alpha_3$. Prove that the area of T is $\alpha_1 + \alpha_2 + \alpha_3 - \pi$.
[Hint: Consider the six wedges made by extending the sides of T in both directions. What regions are multiply covered by the union of the wedges?]

5. Let S be a smooth surface with Gaussian curvature K.

(i) State the local Gauss–Bonnet theorem for a smooth curvilinear polygon $R \subset S$ with boundary γ, n sides and interior angles $\alpha_1, \ldots, \alpha_n$.

(ii) Let $C_{a,b} = \{(x, y, z) : a < x < b, \quad y^2 + z^2 = 1\}$ denote a cylinder of length $b - a$. Suppose that there is a diffeomorphism $f\colon C_{a,b} \to S$ and that, moreover, f is an isometry when restricted to $C_{a,a+\epsilon}$ and $C_{b-\epsilon,b}$ for some small ϵ (i.e, f is an isometry near the ends of the cylinder). Prove that

$$\int_S K \, \mathrm{d}A = 0.$$

[Hint: Let $R' \subset C_{a,b}$ be the four-sided polygon with sides the curves $(x = a + \epsilon/2)$, $(x = b - \epsilon/2)$ and $(y = 1, z = 0)$ twice (once in each direction). Consider the image of R' under f.]