1. (a) State the classification theorem for compact, connected surfaces.

   (b) Let S and S' be compact, connected surfaces. State a formula relating the Euler characteristic of the connected sum $S\#S'$ to the Euler characteristics of S and S'.

   (c) Put the following five surfaces into homeomorphism classes, giving reasons for your answer.

$$T\#K, \quad \mathbb{RP}^2\#K, \quad T\#T, \quad \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2, \quad \mathbb{RP}^2\#T.$$

2. Let $r\colon V \to \mathbb{R}^3$ parametrise a smooth patch of surface.

   (a) (i) Define what it means for r to be *conformal*.

   (ii) State, without proof, a condition on the coefficients of the first fundamental form of r which is equivalent to r being conformal.

   (b) (i) Define what it means for r to be *area preserving*.

   (ii) State, without proof, a condition on the coefficients of the first fundamental form of r which is equivalent to r being area preserving.

   (c) Explain briefly why there does not exist a parametrisation of part of the sphere which is simultaneously conformal and area preserving. (You may assume standard results about Gaussian curvature.)

3. Let $u \mapsto (x(u), y(u))$ be a smooth curve in the (x, y)-plane with $y(u) > 0$. The surface of revolution obtained by rotating this curve about the x-axis is parametrised by

$$r(u, v) = (x(u), y(u)\cos v, y(u)\sin v)\,.$$

Assume that the curve is parametrised by arc-length, so that $x'^2 + y'^2 = 1$.

   (a) Show that the first fundamental form of the surface of revolution is $\mathrm{d}u^2 + y^2\,\mathrm{d}v^2$.

   (b) Show that the second fundamental form of this surface is $(x''y' - x'y'')\,\mathrm{d}u^2 + x'y\,\mathrm{d}v^2$.

   (c) Show that the Gaussian curvature of this surface is $-y''/y$. (You may need to differentiate the condition $x'^2 + y'^2 = 1$ to obtain this.)

4. (a) State the existence and uniqueness theorem for geodesics.

   (b) The ellipsoid is the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$$

   where a, b and c are non-zero. You may assume this is a smooth surface.

   By a considering reflection in the plane $x = 0$, or otherwise, show that the curve $\{(x, y, z) \in S : x = 0\}$ is a geodesic.

5. (a) State the Gauss–Bonnet theorem for a smooth, connected, compact surface in $\mathbb{R}^3$.

   (b) Let $S \subset \mathbb{R}^3$ be a smooth surface homeomorphic to a torus. Prove that the Gaussian curvature of S must vanish at some point.