UNIVERSITY OF LONDON

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| Course: | M3P2/M4P2 |
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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

# M3P2/M4P2

# Measure & Integration

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

# M3P2/M4P2

# Measure & Integration

Date: Tuesday, 22 May 2007 Time: 10.00 am – 12.00 noon

Answer all the questions.

Credit will be given for all questions attempted

but extra credit will be given for complete

or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Candidates should write their solutions in a single answer book.
Supplementary answer books should be used as necessary.

Affix one of the labels provided to each answer book that you use.
DO NOT use the label with your name on it.

1. (i) Define what it means for a subset of $R$ to be, *Basic*, *Outer*, *Inner* and *Integrable*.

(ii) Define the measure of a basic set,and the measure of an outer set.

(iii) Prove that a countable set is outer and has measure zero

(iv) Let $a \in R$ and let $E$ be an integrable set. Prove that the set

$$a + E = \{a + e : e \in E\}$$

is integrable and that $\mu(E) = \mu(a + E)$.

(v)Describe the construction of Cantors Middle Thirds set and prove that it is an inner set with zero measure.

2. (i) Define what it means for a function $f : R \longrightarrow R$ to be upper. Define $\int_R f$ for an upper function, $f$.

(ii) Let $f$ and $g$ be upper functions. Prove that $f \wedge g$ is an upper function.

(iii) Prove that a non-negative continuous function is an upper function.

(iv) Give an example of a real valued function of a real variable which is *not* an upper function.

3. (i) Define what it means for a function to be *measurable*.

(ii)Using any of the theorems proved in the course, which you should clearly state, prove that a monotone increasing function is measurable.

(iii) Let $h$ be a non-negative *measurable* function and $F$ an *integrable* set. Suppose that $|h| \leq M$ on $F$. Prove that $hI_F$ is integrable. You can assume that if $g$ is *integrable* and a set $F$ is *integrable* then $gI_F$ is integrable.

(iv) State the Dominated Convergence Theorem and use it to prove the following: A sequence of non-negative measurable functions, $(f_n)$ converge pointwise to the function $f$ on the integrable set $E$. And, there is a strictly positive number $M$ such that

$$\forall n \in N, |f_n| \leq M$$

on the set $E$. Then $fI_E$ is integrable

4. (i)State the monotone convergence theorem.
(ii) Confirm that for $0 < x < 1$,we have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty} x^{2n}(1-x)$$

(iii) Deduce that

$$\log 2 = \int_0^1 \frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(iv) Let $(f_n)$ be a sequence of non-negative integrable functions for which

$$\sup_k \inf\{\int_R f_n : n \geq k\} < \infty.$$

For each natural number $k$, define $g_k(x) = inf\{f_n(x) : n \geq k\}$. Assuming that the minimum of two integrable functions is itself an integrable function, prove that $g_k$ is an integrable function. Deduce that

$$\int_R g_k \leq \inf\{\int_R f_r : r \geq k\}$$

and that

$$\int_R \lim_{k\to\infty} g_k \leq \lim_{k\to\infty} inf\{\int_R f_n : n \geq k\}.$$

5. (i) State the dominated convergence theorem.
(ii) Let $0 < \theta \leq \pi$ and $\frac{1}{2} \leq r \leq 1$. Show that

$$r^2 + (1-2r)\cos\theta \geq 0.$$

(iii) Deduce from this that

$$\frac{\theta^2(1-r^2)}{1-2r\cos\theta+r^2} \leq \frac{(1-r^2)\theta^2}{1-\cos\theta}.$$

(iv) Prove that the function

$$\frac{(1-r^2)\theta^2}{1-\cos\theta} I_{(0,\pi]}(\theta)$$

is integrable.
(v) Let $(s_n)$ be a sequence of real numbers with $\frac{1}{2} \leq s_n \leq 1$ and $s_n \longrightarrow 1$ as $n \longrightarrow \infty$. Prove that

$$\lim_{n\longrightarrow\infty} \int_0^\pi \frac{\theta^2(1-s_n^2)}{1-2s_n\cos\theta+s_n^2}\, d\theta = 0.$$