Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P2/M4P2

Measure and Integration

Date: Tuesday, 30th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let M be an arbitrary non-empty set, R be a ring of subsets of M, and μ be a measure on R.

(a) Define what it means that R is a ring and μ is a measure on R.

(b) Prove that, for any sequence $\{A_n\}_{n=1}^{\infty} \subset R$, the conditions

$$A_n \subset A_{n+1} \quad \text{and} \quad A := \bigcup_{n=1}^{\infty} A_n \in R$$

imply

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

(c) Let $A \in R$ and $\{A_k\}_{k=1}^{\infty}$ be a sequence of sets from R. Prove that

$$A \subset \bigcup_{k=1}^{\infty} A_k \implies \mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

2. Let R be an algebra of subsets of a set M and μ be a finite measure on R.

(a) Define what it means that R is an algebra. Define the associated *outer measure* μ^*.

(b) Prove that the outer measure μ^* is σ-subadditive.

(c) Prove that if $A \in R$ then $\mu^*(A) = \mu(A)$.

(d) For any set $A \subset M$, define its *inner measure* by

$$\mu_*(A) = \mu(M) - \mu^*(M \setminus A).$$

Prove that $\mu_*(A) \leq \mu^*(A)$. Prove also that if $A \in R$ then $\mu_*(A) = \mu(A)$.

3. Let $\mathcal{M}$ be a σ-algebra of subsets of a set M and μ be a σ-finite complete measure on $\mathcal{M}$.

(a) Define what is means that $\mathcal{M}$ is a σ-algebra and that μ is complete. Define what it means that a real-valued function f on M is *measurable* (with respect to $\mathcal{M}$).

(b) A function f on M is called *elementary* if it can be represented in the form

$$f = \sum_{k=1}^{l} \alpha_k 1_{A_k}$$

where $\{\alpha_k\}_{k=1}^{l}$ is a finite sequence of reals and $\{A_k\}_{k=1}^{l}$ is a sequence of disjoint measurable subsets of M, and 1_A denotes the indicator function of the set A. Prove that any measurable function f on M can be represented as a pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of elementary functions.

(c) Let f be a non-negative measurable function on M. Prove that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative elementary functions on M such that $f_n \to f$ as $n \to \infty$ pointwise and

$$\int_M f_n d\mu \to \int_M f d\mu.$$

State clearly all the results used.

4. (a) State and prove the Dominated Convergence Theorem. State clearly all results used.

(b) Prove that, for any $t > 0$,

$$\frac{d}{dt} \int_0^{\infty} e^{-t\lambda} \frac{\sin \lambda}{\lambda} d\lambda = -\int_0^{\infty} e^{-t\lambda} \sin \lambda d\lambda.$$

Hint: Use the Dominated Convergence Theorem and the inequality

$$|e^x - 1| \leq |x| \, e^{|x|},$$

which is true for all real x.

(c) Using the fact that

$$\int_0^{\infty} e^{-t\lambda} \sin \lambda \, d\lambda = \frac{1}{1+t^2} \, ,$$

evaluate the integral

$$\int_0^{\infty} e^{-t\lambda} \frac{\sin \lambda}{\lambda} d\lambda.$$

5. Let μ be a complete finite measure defined on a σ-algebra $\mathcal{M}$ on a set M.

(a) Define the space $L^1(M,\mu)$ and state the theorem about the completeness of this space.

(b) For any two sets $A, B \in \mathcal{M}$, define the pseudodistance $d(A,B)$ by

$$d(A,B) = \mu(A\Delta B),$$

where $A\Delta B$ is the symmetric difference of the sets A, B. Prove that

$$d(A,B) = \|1_A - 1_B\|_1.$$

(c) Prove the triangle inequality: for any three sets $A, B, C \in \mathcal{M}$,

$$d(A,B) \le d(A,C) + d(C,B).$$

(d) A sequence $\{A_n\}_{n=1}^{\infty}$ of sets from $\mathcal{M}$ is called Cauchy if $d(A_n, A_k) \to 0$ as $n, k \to \infty$. Prove that the family $\mathcal{M}$ is complete with respect to the pseudodistance d in the following sense: any Cauchy sequence $\{A_n\} \subset \mathcal{M}$ converges, that is, there is $A \in \mathcal{M}$ such that $d(A_n, A) \to 0$ as $n \to \infty$.