UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS) 2005

This paper is also taken for the relevant examination for the Associateship.

MEASURE AND INTEGRATION THEORY[M3/4P2 2005]

Answer FIVE questions. Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

The following standard notation is used

- λ = the Lebesgue measure;
- $L - \int$ = the Lebesgue integral;
- $\chi(A)$ = the characteristic function of a set A;
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denotes the set of natural, integer, rational and real numbers, respectively;
- $\mathbf{C}_1$ = the ternary Cantor set.

PLEASE TURN OVER

1. (i) Give the definition of the Borel sigma algebra in $\mathbb{R}^2$. Prove that the smallest σ-algebra containing the family of all open triangles in $\mathbb{R}^2$ is equal to the Borel σ-algebra in this space.

   (ii) Give the definition of a Lebesgue measurable set in $\mathbb{R}$.
   Prove or disprove that the ternary Cantor set in $[0,1]$ is Lebesgue measurable.

2. (i) Prove or disprove that the following functions are Lebesgue integrable and/or Riemann integrable.
   (a) $\chi([0,\frac{1}{2}])\sum_{k\in\mathbb{N}} x^k \sin(k^2\cdot x)$
   (b) $\chi(\mathbb{Q})\frac{\sin x}{x}$ on the interval $(-\infty,\infty)$.

   (ii) Prove or disprove the following statements. Full credit will only be given if the assumptions of any theorems used are verified.
   (a) $\lim_{n\to\infty}\int_0^1 x^{-\frac{1}{n}}\sin^2\left(x^{\frac{n}{2}}\right)\, d\lambda = 1;$
   (b) $\lim_{n\to\infty}\int_{-\infty}^{\infty}\left(1+\frac{x^2}{n}\right)^{-n} e^{-2x-1}\chi(\mathbf{C}_1)\, d\lambda = \sqrt{2\pi};$
   (c) $\lim_{n\to\infty}\int_0^{\infty}\cos\left(2\pi\exp\frac{-x}{n}\right)\exp\left(-x+\sin\left\{\frac{x^4}{2\sqrt{n}}[1+\frac{1}{1+x^3n}]\right\}\right) d\lambda = 1.$

3. (i) State the monotone convergence theorem.
   Prove that the following function

$$A\to\int_A\frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}d\lambda$$

   is a measure on Lebesgue measurable sets.
   Prove or disprove that this measure is absolutely continuous with respect to the Lebesgue measure.

   (ii) State and prove Fatou's Lemma.

4. (i) Define the variation of a function on a bounded interval $[a,b]$.
   Prove that any bounded monotone function on $[a,b]$ has finite variation.
   Prove or disprove that a function with finite variation on $[a,b]$ can be represented as a linear combination of nondecreasing functions.

   (ii) Which of the following functions is absolutely continuous and which is not ? Full credit will only be given if reasons are provided.
   (a) $f(x) = L - \int_{[0,x)}\chi([0,1]\setminus\mathbb{Q})\sin(\pi x^3)d\lambda$ on $[0,1]$;
   (b) $f(x) = \begin{cases} x\cos(1/x^2) & \text{if } x\in(0,1] \\ 0 & \text{if } x=0 \end{cases}$;
   (c) $f(x) = L - \int_{[0,x)} s^{-1/2}\cos(1/s^2)\, d\lambda$

5. Which of the following statements are true and which are not ? No reasoning need be given: just answer T or F in the box next to each question. The marking will be as follows (subject to a minimum score of zero) : each correct answer scores +1, each wrong answer -1 and zero for no answer.

[1] ☐ every sigma algebra contains the union of any sub-family of sets;

[2] ☐ the family of open sets in $\mathbb{R}^d$ is an algebra but not a sigma algebra;

[3] ☐ not every set of measure zero is Lebesgue measurable;

[4] ☐ an outer measure is sigma sub-additive;

[5] ☐ every open subset of $\mathbb{R}^n$ is Lebesgue measurable;

[6] ☐ the characteristic function of the intersection of Vitali set and the Cantor ternary set is not Lebesgue measurable;

[7] ☐ the Lebesgue integral is well defined only for continuous function on $\mathbb{R}$;

[8] ☐ if a sequence of Lebesgue integrable functions on $\mathbb{R}$ converges almost everywhere to a continuous function, its limit is Riemann integrable;

[9] ☐ each function of finite variation on [0,1] is Riemann integrable;

[10] ☐ there exists a function on $[0,1]$ of infinite variation which is Lebesgue integrable;

[11] ☐ if a measurable function f has bounded derivative on $[a,b]$ then $L-\int_{[a,x)} \frac{df}{dt}\, dt = f(x) - f(a)$ for any $x \in [a,b]$;

[12] ☐ the function $\frac{\sin x}{x}$ is Lebesgue integrable on $\mathbb{R}$;

[13] ☐ the family of functions which are Lebesgue square integrable form a linear space;

[14] ☐ the monotone convergence theorem is true for all positive as well as all negative functions;

[15] ☐ the integral of a sum of infinite number of nonnegative functions equals the sum of the integrals of these functions;

[16] ☐ if a function is λ-square integrable on $\mathbb{R}$, then it is also λ-integrable;

[17] ☐ all functions on a bounded open interval which are differentiable, are also Riemann integrable;

[18] ☐ if a function is not differentiable almost everywhere on a finite interval, its variation is infinite;

[19] ☐ if a sequence of measurable functions converges in the 2nd power, it converges almost everywhere;

[20] ☐ if a sequence of measurable functions converges uniformly on a real line set, it converges in 2nd power.

THE END