M3P17 exam 2007

1. Define what is meant by a linear code of length n, dimension k, which corrects e errors.

State (but do not prove) the Hamming bound and the Gilbert-Varshamov bound.

For each of the following statements, say whether it is true or false, justifying your answer. State any standard results you use.

(i) There exists a linear code of length 20 and dimension 13 which corrects 2 errors.

(ii) There exists a linear code of length 20 and dimension 9 which corrects 2 errors.

(iii) There exists a linear code of length 10 and dimension 3 which corrects 2 errors.

(iv) There exists a linear code of length 10 and dimension 2 which corrects 3 errors.

2. Define the following:

an e-perfect code of length n

a t-design.

Suppose that C is a linear code of length 24, dimension 12, and minimum distance 8.

(a) Let C' be the code of length 23 consiting of all the codewords in C with their last digit deleted. Prove that C' is 3-perfect.

(b) Let X be the set of 24 coordinate positions in $\mathbb{Z}_2^{24}$, and for a codeword $c \in C$ of weight 8, define B_c to be the subset of X consisting of the positions of the eight 1's in c. Define $\mathcal{B}$ to be the collection of all such subsets B_c $(c \in C)$. Prove that $\mathcal{B}$ is a 5-design.

(c) Deduce that the number of codewords in C of weight 8 is equal to 759.

3. Let Γ be a graph with a finite set of vertices. Assume that any two vertices joined by an edge have no common neighbours, and any two vertices not joined by an edge have exactly 1 common neighbour. Assume also that there is no vertex which is joined to all the others.

(i) Prove that Γ is regular.

(ii) Let k be the valency of Γ, and let v be the number of vertices. Prove that $v = k^2 + 1$.

(iii) Gives examples of such a graph Γ for $k = 2$ and $k = 3$.

4. Define what is meant by a strongly regular graph with parameters (v, k, a, b).

Let Γ be a strongly regular graph with parameters $(v, k, 0, 3)$, and assume $k > 3$. Stating any standard results you require, prove the following.

(i) $v = \frac{1}{3}(k^2 + 2k + 3)$.

(ii) $4k - 3$ is a square (of an integer).

(iii) $\sqrt{4k-3}$ divides k^2.

(iv) $k = 21$.

5. Let $V = \mathbb{Z}_2^n$, a vector space of dimension n over $\mathbb{Z}_2$. Assume that $n \geq 3$.

(i) State and prove a formula for the number of 2-dimensional subspaces of V.

(ii) Define a design as follows: the points are the vectors in V, and the blocks are all subsets of the form $v + W$, where $v \in V$ and W is a 2-dimensional subspace of V. (Recall that $v + W = \{v + w \, : \, w \in W\}$.)

Prove that these blocks form a 3-design, and find its parameters.

(iii) Calculate the total number of blocks of the design in part (ii). Given a pair of vectors $v, w \in V$, how many blocks are there containing both v and w ?

(iv) Prove that the design in part (ii) is not a 4-design.