1. State a criterion in terms of its minimal polynomial for an algebraic number to be an algebraic integer.

   Let $d$ be a square-free integer not equal to 1. Determine, with proof, the ring of algebraic integers in $\mathbb{Q}(\sqrt{d})$.

2. Prove that the class group of $\mathbb{Q}(\sqrt{-23})$ is cyclic of order 3.

   *You may assume any theorems from the course that you need.*

3. Let $K$ be a quadratic number field, and let $R$ be its ring of integers.

   (i) Define the norm $||I||$ of an ideal $I$ of $R$.

   (ii) State, without proof, a result relating the norm of the principal ideal $(\alpha)$, $\alpha \in R$, to the norm of the element $\alpha$.

   (iii) Prove that if $R$ contains a unit of norm $-1$, then it contains infinitely many such units.

   (iv) Let $K = \mathbb{Q}(\sqrt{14})$. Determine, with proof, whether $R$ contains a unit of norm $-1$.

4. Find all solutions to the equation $y^2 = x^3 - 13$ with $x, y \in \mathbb{Z}$.

   *You may assume that the class number of $\mathbb{Q}(\sqrt{-13})$ is 2, and that the ring of integers in $\mathbb{Q}(\sqrt{-13})$ is $\mathbb{Z}[\sqrt{-13}]$.*

5. Let $K$ be a number field, and $R$ its ring of integers. Prove that if $A$, $B$, and $C$ are non-zero ideals of $R$ such that $AB = AC$, then $B = C$.

   *You may assume that for any proper ideal $I$ of $R$ there exists a $\gamma \in K$, $\gamma \notin R$, such that $\gamma I \subset R$. You should not assume that every ideal is a product of prime ideals.*