1. (a) Give definitions of the following:

   (i) An *algebraic number*.

   (ii) An *algebraic integer*.

   (iii) A *number field*.

   (b) Prove from the definition that $1 + \sqrt{2} + \sqrt{3}$ is an algebraic integer.

   (c) Prove that if K is a number field and $\alpha \in K$, then α is an algebraic number.

   (d) If K is a number field, and $\alpha \in K$, define the *norm* $N_{K/\mathbb{Q}}(\alpha)$ and the *trace* $T_{K/\mathbb{Q}}(\alpha)$.

   (e) Prove that if K is a number field and α, $\beta \in K$ then $N_{K/\mathbb{Q}}(\alpha\beta) = N_{K/\mathbb{Q}}(\alpha)N_{K/\mathbb{Q}}(\beta)$ and $T_{K/\mathbb{Q}}(\alpha + \beta) = T_{K/\mathbb{Q}}(\alpha) + T_{K/\mathbb{Q}}(\beta)$.

2. Let $d \neq 1$ be a squarefree integer, and let $K = \mathbb{Q}(\sqrt{d})$. State a criterion in terms of $N_{K/\mathbb{Q}}(\alpha)$ and $T_{K/\mathbb{Q}}(\alpha)$ for $\alpha \in K$ to be an algebraic integer and find, with proof, the algebraic integers of K.

3. Let K be a number field, and R its ring of integers, and let P be a nonzero prime ideal of R. Prove that P is a maximal ideal. *Hint*: You may assume that a finite integral domain is a field, and that R is a free abelian group of rank n, where n is the degree of K.

4. Show that the class number of $\mathbb{Q}(\sqrt{-6})$ is 2, given the information that every ideal class contains an integral ideal I with $||I|| < 4$. You may assume standard results on the structure of the ring of integers of a number field.

5. Show that the only solutions of the Diophantine equation

$$y^2 + 11 = x^3 \quad (x, y \in \mathbb{Z})$$

are $(x, y) = (3, \pm 4), (15, \pm 58)$.*Hint*: You may assume that the class number of $\mathbb{Q}(\sqrt{-11})$ is 1, that the ring of integers in $\mathbb{Q}(\sqrt{-11})$ is $\mathbb{Z}[(1 + \sqrt{-11})/2]$, and that the only units in $\mathbb{Z}[(1 + \sqrt{-11})/2]$ are ± 1.