1. (i) If u, v are positive integers, hcf$(u, v) = 1$, and uv is a perfect square, show that u and v are both perfect squares.

(ii) Let x, y, z be positive integers with no common divisors such that

$$x^2 + y^2 = z^2.$$

Show that either x or y is even. Assume now that x is even. Prove that there exist positive integers u, v with hcf$(u, v) = 1$ and such that $z - y = 2u^2$, $z + y = 2v^2$. Hence, show that $x = 2uv$, $y = v^2 - u^2$ and $z = v^2 + u^2$. Conversely, if u, v are given, show that the numbers x, y and z given by these formulas satisfy $x^2 + y^2 = z^2$.

2. In this question, you may use without proof the following facts: There are constants $\gamma > 0$ and $A > 0$ such that

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\Big(\frac{1}{x}\Big), \text{ and}$$

$$\sum_{n \le x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + A + O\Big(\frac{\log x}{x}\Big).$$

For $n \ge 1$ integer, denote by $d(n) = \sum_{d|n} 1$ the number of divisors of n. Prove that

$$\sum_{n \le x} \frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma \log x + O(1).$$

3. In this question, you may assume without proof any properties of the Legendre symbol, as long as you state them correctly. State and prove a condition characterising those primes p which can be expressed in the form

$$p = x^2 + 2y^2$$

with x, y integers.

4. In the ring $\mathbb{Z}[i]$ of Gaussian integers, we say $a \equiv b$ mod m if m divides $a - b$.

(i) Find a Gaussian integer x such that

$$(1+2i)x \equiv 1 \mod 3+3i$$

In what sense is your solution unique? (Justify your answer.)

(ii) Find a Gaussian integer x such that

$$\begin{aligned} x &\equiv 1 \mod 1+2i \\ x &\equiv 3 \mod 3+3i. \end{aligned}$$

In what sense is your solution unique? (Justify your answer.)

5. Let $\alpha = \sqrt{2} + \sqrt{3}$. In this question, you may assume that

$$f(x) = x^4 - 10x^2 + 1$$

is a rational polynomial of minimal degree such that $f(\alpha) = 0$.

(i) Find a polynomial $g(x)$ such that $f(x) = (x - \alpha)g(x)$.

(ii) Find an explicit constant K such that if p/q is any rational number with $|(p/q) - \alpha| \leq 1$, then

$$\left| g\left(\frac{p}{q}\right) \right| \leq K.$$

(iii) Find an explicit constant C such that for all rational numbers p/q

$$\left| \frac{p}{q} - \alpha \right| > \frac{C}{q^4}.$$

(Justify your answer.)