Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P13/M4P13

Rings and Modules

Date: Wednesday, 24th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

In this paper, R denotes an arbitrary ring, possibly noncommutative, unless otherwise stated. All modules are taken to be left modules.

1. Define

   (i) a *simple ring*;

   (ii) a *ring homomorphism* $\theta : R \to S$.

   Verify that the kernel $\mathrm{Ker}(\theta)$ is a twosided ideal in R.

   Is $\mathrm{Im}(\theta)$ a twosided ideal in S?

   Show that if R is simple and $S \neq 0$, then any ring homomorphism $\theta : R \to S$ must be injective.

   Now let F be a field and let $R = M_2(F)$ be the ring of 2×2 matrices over F. Show that R is simple.

   Find a ring $S \neq R$ so that there is a ring homomorphism $R \to S$, defining your homomorphism explicitly.

2. Let M and N be R-modules. Say what is meant by an R*-module homomorphism* $\theta : M \to N$.

   Let L be a submodule of M. Define the quotient module M/L, giving the addition and scalar multiplication explicitly. (You are not expected to verify the module axioms.)

   Show that there is an injective induced homomorphism

$$\overline{\theta} : M/\mathrm{Ker}(\theta) \to N \ \text{ with } \ \overline{\theta}\pi = \theta$$

   – you are expected to verify that your homomorphism is well-defined and injective.

   Suppose that M has two distinct maximal submodules L, P. Show that there is an R-module isomorphism

$$L/L \cap P \cong M/P.$$

   Now let $T = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$ be the ring of 2×2 upper triangular matrices over a field F. Find two distinct maximal left ideals H, J of T, and identify the factor modules T/H and T/J – you should say how an element of T acts on T/H and on T/J.

3. Define the terms

(i) A *Noetherian* left R-module;

(ii) An *Artinian* left R-module.

State without proof two alternative characterizations of a Noetherian module.

Let M be a left R-module with submodule L and put $N = M/L$. Show that M is Noetherian if and only if both L and N are Noetherian.

Give (with reasoning) examples of

(a) A ring that is neither left Artinian nor left Noetherian.

(b) A ring that is not Artinian that has a nonzero Artinian module.

(c) A ring that is Artinian that has a non-Artinian module.

(d) A Noetherian ring that has a non-Noetherian Artinian module.

4. In this question, you are not expected to check the ring or module axioms when you claim that something is a ring or module.

Let $R_1, \ldots, R_n$ be a finite set of rings. Define their *direct product* $R = R_1 \times \cdots \times R_n$, giving the addition and multiplication. Show that $R = H_1 \oplus \cdots \oplus H_n$ where each H_i is both a twosided ideal of R and a ring, and $R_i \cong H_i$ as a ring.

Let M be a left R-module. Show that $M = M_1 \oplus \cdots \oplus M_n$ where each M_i is an R_i-module.

Explain how a set $M_1, \ldots, M_n$ with each M_i an R_i-module gives rise to an R-module.

Let $k \geq 1$ be an integer. Give an example of a ring R that has a composition series (as left R-module) of length k, all of whose composition factors are isomorphic – a proof is not required, but you should write down the composition series.

Give, with proof, an example of a ring S with a composition series of length $k \geq 1$ such that no two composition factors are isomorphic as S-modules.

5. Let R be a commutative ring. Define the following terms.

(i) A *prime* ideal P of R.

(ii) A *multiplicatively closed* subset of R.

(iii) The *nilradical* $\mathrm{Nil}(R)$ of R.

Verify that $\mathrm{Nil}(R)$ is an ideal of R, and compute $\mathrm{Nil}(R/\mathrm{Nil}(R))$.

Show that the complement $R \setminus P$ of a prime ideal P is multiplicatively closed.

Let S be multiplicatively closed. Show further that if I is an ideal which is maximal among the set Σ of ideals in R with $I \cap S = \emptyset$, then I is prime.

Deduce that $\mathrm{Nil}(R) = \bigcap\{P \mid P \text{ is a prime ideal}\}$.

Now let F be a field and put $S = F[\sigma, \tau]$, with $\sigma^2 = \tau^2 = s^2\tau^2 = 0$. (In other words, $S = F[X,Y]/X^2F[X,Y] + Y^2F[X,Y] + (XY)^2F[X,Y]$.) Let $R = S \times S$, the direct product of rings.

Find $\mathrm{Nil}(S)$, and hence find $\mathrm{Nil}(R)$ and $R/\mathrm{Nil}(R)$.