UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

M3P12

Representation theory

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Representation theory

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $(\rho_1, \mathbb{C})$ be the trivial representation of S_4, and let χ_1 be its character.

   (i) Let V be the complex vector space with basis x_1, x_2, x_3, x_4, and let (τ, V) be the representation of S_4 determined by

   $$\tau(\sigma).x_i = x_{\sigma(i)}$$

   for $i \in \{1, \ldots, 4\}$ and $\sigma \in S_4$. Let χ_τ be the character of ρ. Show that for all $\sigma \in S_4$, we have

   $$\chi_\tau(\sigma) = | \{i \in \{1, 2, 3, 4\} : \sigma(i) = i\} | .$$

   (ii) Show that $(\tau, V) = (\rho_1, \mathbb{C}) \oplus (\rho_2, W)$ for some irreducible representation (ρ_2, W).

   (iii) Define the sign representation $(\rho_3, \mathbb{C})$ of S_4 and calculate its character.

   (iv) Show that the representation (ρ_4, W) defined by

   $$\rho_4(g).v = \rho_3(g)\rho_2(g).v$$

   is irreducible.

   Remark. You may assume that two elements in S_4 lie in the same conjugacy class if and only if they are of the same cycle type.

2. (i) Show from first principles that every irreducible representation of the cyclic group C_n of order n is 1-dimensional.

   (ii) Decompose the regular representation of C_n explicitly as the sum of 1-dimensional representations.

3. Let G be a finite group and $\{\chi_i\}$ the set of its irreducible characters. Choose representatives g_j of the conjugacy classes of G, and denote by $Z(g_j)$ their centralizers.

(i) State the orthonormality and completeness relations for the χ_i.

(ii) Using part (i), show that

$$\sum_i \overline{\chi_i(g_j)}\chi_i(g_k) = \begin{cases} \mid Z(g_j) \mid & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

(iii) Let A be the matrix with $A_{ij} = \chi_i(g_j)$. Prove that

$$\mid \det A \mid^2 = \prod_j \mid Z(g_j) \mid .$$

Hint. Note that $\mid \det A \mid^2 = \overline{\det A}.\det A$ and that $\det A = \det A^t$, where A^t denotes the transpose of A.

4. Let (ρ, V) be an irreducible representation of a group G.

(i) Define a G-homomorphism $V \to V$ and state and prove Schur's lemma.

(ii) Let Z be the center of G. Using Schur's Lemma, show that if $g \in Z$, then $\rho(g)$ is a scalar multiple of the identity.

(iii) Deduce that if G is commutative, then V is 1-dimensional.

5. Let G be a finite group, and let $H \leq G$ be a subgroup. Let V be the vector space whose basis vectors are the elements of G/H. Define an action of G on V by $g.xH = (gx)H$.

(i) Let χ_ρ by the character of the representation. Show that $\chi_\rho(g) = \mid \{xH : g \in xHx^{-1}\} \mid$.

(ii) Show that if H normal in G, then $\chi_\rho(g) = [G : H]$ if $g \in H$ and 0 otherwise.

(iii) Suppose that $H \neq G$. Show that V is not irreducible.