1. (a) Let G be a group. Define the notions of $\mathbb{C}G$-module, $\mathbb{C}G$-submodule. State (without proof) Maschke's theorem.

(b) Give a counterexample (with an explanation) to the statement of Maschke's theorem in the case of an infinite group.

(c) Let $G = C_2 \times C_2$ be a direct product of two cyclic groups of order two, let a and b be the generators of these cyclic groups.

(i) Write down (without proof) a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules $V_1, \ldots, V_k$.

(ii) Consider a 4-dimensional representation ρ of G given by

$$\rho(a) = \begin{pmatrix} 5 & -4 & 0 & 0 \\ 6 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Decompose the corresponding $\mathbb{C}G$-module into a direct sum of irreducible $\mathbb{C}G$-submodules $U_1, \ldots, U_l$, and justify your answer. For each submodule U_i find an irreducible V_j from Part (i) which is $\mathbb{C}G$-isomorphic to it, with justification.

2. Let G be a finite group, let U and V be $\mathbb{C}G$-modules.

(i) Define the notion of $\mathbb{C}G$-homomorphism from U to V, define the vector space structure on the set $\mathrm{Hom}_{\mathbb{C}G}(U, V)$ of all $\mathbb{C}G$-homomorphisms from U to V.

(ii) Prove the version of Schur's lemma which states that any $\mathbb{C}G$-homomorphism from an irreducible $\mathbb{C}G$-module to itself is a multiplication by a constant.

(iii) Let U be an irreducible $\mathbb{C}G$-module, let $V = U_1 \oplus U_2$ where the $\mathbb{C}G$-submodules U_1, U_2 are both $\mathbb{C}G$-isomorphic to U. Determine a basis of the space $\mathrm{Hom}_{\mathbb{C}G}(U, V)$ and find the dimension of this space. Present arguments to justify your answers.

3. Let G be a finite group.

(a) (i) Define the regular $\mathbb{C}G$-module, and write down the values of its character χ_{reg}. Justify your answer.

(ii) Let $V_1, \dots, V_k$ be $\mathbb{C}G$-submodules of $\mathbb{C}G$ which form a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules, let $f_1, \dots, f_k$ be the corresponding primitive central idempotents. Prove the formula

$$f_i = \frac{\chi_i(e)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g,$$

where χ_i is the character of the module V_i, $i = 1, \dots, k$. You can use without a proof the property that for all $i, j \leq k$,

$$\rho_i(f_j) = \begin{cases} I_{\chi_i(e)}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

where $I_{\chi_i(e)}$ stands for the identity matrix of size $\chi_i(e)$, and ρ_i is the linear extension to $\mathbb{C}G$ of a representation corresponding to the module V_i. You can also use without a proof that $\chi_{reg} = \sum_{i=1}^{k} \chi_i(e)\chi_i$.

(b) Assume now that $k = 2$ and consider a basis h_1, h_2 of the centre $Z(\mathbb{C}G)$ such that

$$h_1^2 = h_1, \; h_2^2 = h_2, \; h_1 h_2 = 0.$$

Prove that the set $\{h_1, h_2\}$ is equal to the set $\{f_1, f_2\}$. (You may assume that the primitive central idempotents form a basis of $Z(\mathbb{C}G)$ which satisfies $f_1^2 = f_1, f_2^2 = f_2, f_1 f_2 = 0$.)

4. (a) Define the inner product $\langle \cdot, \cdot \rangle$ on the characters of a finite group G and state (without a proof) a result giving the possible values of $\langle \varphi, \psi \rangle$ for irreducible characters φ, ψ. Deduce that if a character Φ has a decomposition

$$\Phi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where $\chi_1, \ldots, \chi_k$ are different irreducible characters then $\lambda_i = \langle \Phi, \chi_i \rangle$.

(b) The symmetric group S_4 has the following character table:

Conjugacy class:	e	(12)	(123)	$(12)(34)$	(1234)
Centralizer order:	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0

(i) Express the character Φ given by the following values

$$\Phi(e) = 10, \Phi((12)) = 0, \Phi((123)) = 1, \Phi((12)(34)) = -2, \Phi((1234)) = 2$$

as a linear combination of the irreducible characters.

(ii) Express the tensor square character $\chi = \chi_4^2$, as well as the antisymmetric part χ_a of the character χ and the symmetric part χ_s of χ, as linear combinations of the irreducible characters.

5. (a) Let G be a finite group, g an element of G of order m, and χ a character of G of degree n. Prove that $\chi(g)$ is a sum of n m^{th} roots of unity (you can use without a proof the result on dimensions of irreducible modules of an abelian group).

(b) A group G of order 12 is known to have an irreducible character which is non-zero only on two conjugacy classes where it equals 3 and -1. Making it clear which properties of the characters you use,

(i) Determine the number of conjugacy classes of G and the degrees of the irreducible characters.

(ii) Fill in any two rows and any two columns of the character table of G.

(iii) Complete the character table.