1. Show that the equation:

$$x^5 - 9x + 3 = 0$$

cannot be solved in radicals over $\mathbb{Q}$.

(You may use any theorem you heard during the lectures.)

2. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $p(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$ and let $K = \mathbb{Q}(\alpha)$.

(i) Show that $(\alpha^2 - 2)^2 - 2 = -\alpha^2 - \alpha + 1$.

(ii) Prove that $\alpha^2 - 2$ is also a root of $p(x)$.

(iii) Show that $-\alpha^2 - \alpha + 1$ is also a root of $p(x)$.

(iv) Show that $[K : \mathbb{Q}] = 3$.

(v) Prove that $K|\mathbb{Q}$ is a Galois extension.

(vi) Determine the Galois group of the extension $K|\mathbb{Q}$.

3. Let $p(x) \in K[x]$ be a polynomial of degree n. Show that $[L : K]$ divides $n!$ where $L|K$ is the splitting field of $p(x)$.

4. Prove that for every finite simple extension $L|K$ there are only finitely many subfields $F \subseteq L$ containing K.

5. Prove that there is a field K of characteristic zero which has two subfields $K_1 < K$ and $K_2 < K$ such that the extensions $K|K_1$ and $K|K_2$ are finite but the extension $K|K_1 \cap K_2$ is not finite.

(You may use any theorem proved in the course.)