UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P11/M4P11

Galois Theory

Date: Wednesday 31st May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Notation: $L : K$ denotes an extension of fields with $K \subseteq L$. In particular $K(Y)$ denotes a field K with a set Y adjoined. The Galois group of $L : K$ is written $\Gamma(L : K)$. $*$ and $\dagger$ will denote the maps of the Galois correspondence.

1. (a) Let $L : K$ be an extension of fields. Define the terms

   (i) $L : K$ is simple.

   (ii) $L : K$ is algebraic.

   (iii) $L : K$ is normal.

   (iv) $L : K$ is separable.

   (b) Let $X = \{\sqrt[n]{5} \,|\, n \in \mathbb{N}\}$ and set $F = \mathbb{Q}(X)$. Is the extension $F : \mathbb{Q}$

   (i) algebraic ?

   (ii) simple?

   (iii) normal?

   Justify your answers.

2. (a) (i) Define the Galois group $\Gamma(L : K)$ of an extension of fields $L : K$. For each intermediate field M with $K \subseteq M \subset L$, define $M^* \subseteq \Gamma(L : K)$. For each subgroup $H \subseteq \Gamma(L : K)$, define $H^\dagger$.

   (ii) Let $f(t) = t^3 - 3 \in \mathbb{Q}[t]$. Identify the splitting field Σ for f over $\mathbb{Q}$. Find a presentation for the Galois group $\Gamma(\Sigma : \mathbb{Q})$.

   (b) Draw a diagram illustrating the Galois correspondence for $\Sigma : \mathbb{Q}$. Identify each subgroup and subfield in your diagram.

3. (a) (i) Define a radical extension $R : \mathbb{Q}$, and explain what it means for a polynomial in $\mathbb{Q}[t]$ to be soluble by radicals.

   (ii) Give a brief account of the proof that the Galois group $\Gamma(R : \mathbb{Q})$ is soluble when $R : \mathbb{Q}$ is a normal and radical extension.

   (b) Let Σ be a splitting field for the polynomial $f(t) = t^5 - 9t + 3$ over $\mathbb{Q}$. Show that the Galois group $\Gamma(\Sigma : \mathbb{Q})$ is isomorphic to S_5. Deduce that $f(t)$ is not soluble by radicals.

4. (a) Let Σ be a splitting field for a cubic polynomial over $\mathbb{Q}$ such that $\Gamma(\Sigma : \mathbb{Q})$ is isomorphic to the cyclic group of order three. Prove that the extension $\Sigma : \mathbb{Q}$ is not radical.

   (b) The field $\mathbb{Q}(\cos \pi/9)$ is the splitting field for a cubic polynomial over $\mathbb{Q}$. Find a field R with $\mathbb{Q}(\cos \pi/9) \subset R$ such that the extension $R : \mathbb{Q}$ is radical. Find the Galois group $\Gamma(R : \mathbb{Q})$ of your example.

5. (a) (i) State the theorem of primitive element.

   (ii) Let p and q be distinct primes. Show that the extension $\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}$ is simple.

   (b) Let s be transcendental over $\mathbb{Z}_2$ and t transcendental over $\mathbb{Z}_2(s)$. Consider the extension $L : K = \mathbb{Z}_2(t, s) : \mathbb{Z}_2(t^2, s^2)$.

   (i) Find the degree $[L : K]$.

   (ii) Let $a \in L$. Show that $a^2 \in K$.

   (iii) Show that $L : K$ is not simple. Explain why this does not contradict the theorem of primitive element.