Imperial College London

UNIVERSITY OF LONDON

Course:	M3/4P11
Setter:	Keating
Checker:	James
Editor:	James
External:	Donkin
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3/4P11 Galois Theory

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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Note. In this paper, all fields are taken to be subfields of the complex numbers $\mathbb{C}$.

You may quote results from group theory without proof.

1. Let E/F be an extension of fields. Define the terms

(i) E/F is simple;

(ii) a primitive element of E/F;

(iii) E/F is normal.

Let $a = \sqrt{3 + \sqrt{11}}$. Find the minimal polynomial $f(X)$ of a over $\mathbb{Q}$, and determine b so that the 4 roots of $f(X)$ are $\pm a, \pm b$.

Show that $\mathbb{Q}(a)/\mathbb{Q}$ is not a normal extension. *Hint:* you do not need to attempt to write b as a 'general' element of $\mathbb{Q}(a)$.

Find the splitting field E of $f(X)$ over $\mathbb{Q}$ and the degree $|E : \mathbb{Q}|$.

Show that $a + b$ is *not* a primitive element of $E/\mathbb{Q}$.

Show also that the fields $\mathbb{Q}(a)$, $\mathbb{Q}(b)$ and $\mathbb{Q}(a + b)$ are all different.

2. Define the *Galois group* $G = \mathrm{Gal}(E/F)$ of a normal extension E/F of fields.

Given an intermediate field K, define the subgroup K^* of G.

Given a subgroup H of G, define the intermediate field $H^\dagger$.

State the relationship between K and $K^{*\dagger}$, and between $H^{\dagger *}$ and H.

In the remainder of this question, you may quote without proof any results on cyclotomic polynomials that you wish to use.

Let ω be a primitive 7-th root of unity, and let $E = \mathbb{Q}(\omega)$. Show that $\mathrm{Gal}(E/\mathbb{Q})$ is cyclic of order 6. Find subfields L, M of E with

(a) $|\,\mathrm{Gal}(L/\mathbb{Q})| = 2$,

(b) $|\,\mathrm{Gal}(M/\mathbb{Q})| = 3$,

in each case giving a primitive element.

3. Let F be a field and let $f(X)$ be an irreducible polynomial of degree n over F. Show that the roots $\alpha_1, \ldots, \alpha_n$ of $f(X)$ are distinct. *(You may assume results concerning greatest common divisors of polynomials.)*

Let E be the splitting field of $f(X)$ over F, and let α, β be any two roots of $f(X)$. Show that there is an element $\phi \in \mathrm{Gal}(E/F)$ with $\phi(\alpha) = \beta$.

Put $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)$. Show that $\mathrm{Gal}(f(X)) \subseteq A_n$ if and only if $\Delta \in F$.

4. Explain what is meant by 'the polynomial $f(X)$ is soluble by radicals'.

Let p be a prime number, and let $f(X) = X^5 - 5p^2X + p$, a polynomial over $\mathbb{Q}$. Show that $f(X)$ is not soluble by radicals.

Let $n \geq 5$. Give an example of an irreducible polynomial over $\mathbb{Q}$ of degree n that can be solved by radicals.

5. Let p be a prime number, and suppose that the field K contains a primitive p-th root of unity. Let L/K be a normal extension of degree p. Show that $L = K(\alpha)$ where α is a root of a polynomial $X^p - a$, $a \in K$.