Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3P10/M4P10

Group Theory

Date: Monday, 22nd May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let G be a finite group and let $g \in G$. What is meant by the conjugacy class $Cl(g)$ of g? What is meant by the centralizer $C_G(g)$ of g in G?

   State the relationship between $|Cl(g)|$ and $|C_G(g)|$.

   Suppose that H is a subgroup of G with $|G:H| = 2$. Prove that if $h \in H$ then either $C_G(h) = C_H(h)$ or $|C_G(h)| = 2\,|C_H(h)|$.

   Prove that every conjugacy class of the symmetric group S_n consists of all elements of a given cycle type.

   Find the centralizers in S_5 of the elements (12345) and $(12)(34)$. Find the centralizers in A_5 of the same elements.

2. State what is meant by a simple group. What is meant by a composition series of a group? Prove that every finite group has a composition series, and give an example (with justification) of an infinite group which has no composition series.

   Let $p,\ q$ and r be prime numbers. How many composition series has $C_p \times C_q \times C_r$ in the following cases?

   (a) $p,\ q$ and r are distinct;

   (b) $p = q = r$.

   Justify your answers.

   State and prove the Jordan-Hölder Theorem for finite groups.

3. State Sylow's Theorems. Prove that every finite group G has a Sylow p-subgroup.

   (a) Prove that if H is a normal subgroup of G with $|G:H|$ coprime to p, then H contains every Sylow p-subgroup of G.

   (b) Prove that if H is a normal subgroup of G with $|H| = p^m$ for some integer m, then H is contained in every Sylow p-subgroup of G.

4. Prove that if H is a subgroup of a group G and $|G : H| = n$, then there exists a normal subgroup K of G such that $|G : K|$ divides $n!$ Prove also that K is a subgroup of gHg^{-1} for all $g \in G$.

Prove that there are no simple groups of the following orders: $72,\ 105,\ 1000.$
You may use Sylow's Theorems, but any other result to which you appeal should be proved.

Which results, stated without proof in the course, would be helpful in showing that there are no simple groups of order $72,\ 105$ or 1000?

5. Let G be a finite group and p be a prime number. Define the centre $Z(G)$ of G.

(a) Prove that if $G/Z(G)$ is cyclic then G is abelian.

(b) Prove that if G is a non-trivial p-group then $Z(G)$ is not trivial.

Now let G be a non-abelian group with $|G| = p^4$.

(c) Prove that $|Z(G)| = p$ or p^2.

(d) Prove that if $|Z(G)| = p$ then G contains a conjugacy class of order p.

(e) Prove that if $|Z(G)| = p^2$ then G contains an abelian subgroup of order p^3.