Q1. **(a)** Give a definition of a metric space explaining carefully all notions involved.

(b) Prove or disprove that the following is a metric

(i) $d(x, y) \equiv |f(x) - f(y)|$, where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function;

(ii) $d(g, h) \equiv \int_a^b |g(s) - h(s)|\, ds$ on the space of Riemann integrable functions on a bounded interval $[a, b]$.

(c) **(i)** Give a definition of an open set and a bounded set in a metric space;

(ii) Prove or disprove that the following set is open in a metric space (X, d). $\{x \in X : d(x_0, x) < r\}$, where $x_0 \in X$ and $r > 0$.

Q2. **(a)** Give a definition of a topological space and a Hausdorff space explaining carefully all notions involved.

(b) **(i)** Prove or disprove that every metric space is Hausdorff;

(ii) Give an example of a topological space which is not Hausdorff space.

(c) **(i)** Give a definition of a convergent sequence in a topological space;

(ii) Prove or disprove that the limit is unique in any Hausdorff space.

(d) Prove that if $(X_i, \mathcal{T}_i)$, $i = 1, 2$, are Hausdorff, then the product space is also Hausdorff.

Q3. **(a)** Give a definition of a continuous function between topological spaces using open sets.

(b) **(i)** Give a definition of a closed set in a topological space.

(ii) Prove or disprove that a function $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous iff $C \subset X_1$ closed $\Longrightarrow$ its preimage is closed.

(c) Give a definition of a homemorphism and prove or disprove that the following spaces with euclidian metric are homeomorphic.

(i) $[0, 1] \times [0, 1)$

(ii) $\{(x, y) : -\sqrt{1 - x^2} \leq y < \sqrt{1 - x^2},\ x \in (-1, 1)\ or\ [|x| = 1\ and\ y = 0]\}$

(d) **(i)** Give a definition of the topological invariant.

(ii) Prove or disprove that property of being Hausdorff is a topological invariant

Q4. **(a)** Give a definition of a compact and a sequentially compact space.

(b) Which of the following spaces is compact and which is not. Justify your answer.

(i) $(X, \mathcal{T})$, where X is an infinite set and $\mathcal{T} = \{X, \emptyset, A \subset\ X\ such\ that\ X \setminus A\ is\ finite\}$

(ii) $\mathcal{C} \equiv \{x \in \mathbb{R}^n :\ x \neq \mathbf{0},\ 0 \geq \frac{x}{|x|} \cdot n_1 \geq -\frac{1}{2}\}$ with metric topology, where $|x|$ denotes the euclidian length of the vector x and n_1 is a unit vector.

(c) State and prove Lebesgue Covering Lemma.

(d) Prove or disprove that the following function is bounded $f : [0, 1] \to \mathbb{R}$

$$f(x)\ \equiv \begin{cases} x \cdot \exp\{|\log x|^{\frac{1}{2}}\} & if\ \ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$$

Q5. **(a)** Give a definition of a connected and a path connected space.

(b) Prove or disprove that connectedness is a topological property.

(c) Prove or disprove that there is no continuous injective map of an open unit ball B_1 in the euclidian plane onto the unit interval $(0, 1)$.

(d) Prove or disprove the following

(i) l_2 space is path connected, but not connected;

(ii) a countable union of open intervals is connected, an uncountable union of closed intervals may be not connected.