Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2005

This paper is also taken for the relevant examination for the Associateship.

M3P1/M4P1 Metric and Topological Spaces

Date: Friday 3 June 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Give definitions of a *topology* and of a *topological space*.

(i) Prove that the intersection of any number of topologies on X is again a topology on X.

(ii) Let $\mathcal{A}$ be any collection of subsets of X. Show that there is the smallest topology on X containing all sets from $\mathcal{A}$, that is, there is a topology T on X such that $T \supset \mathcal{A}$ and if S is another topology on X such that $S \supset \mathcal{A}$, then $T \subset S$.

(iii) Define the *product topology*. Show that the product topology on the product $X \times Y$ of topological spaces (X, T) and (Y, S) is the smallest topology for which projections $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ are continuous (projections are defined by $p_X(x, y) = x, p_Y(x, y) = y$).

2. Let X be a topological space. Define what it means for X to be *compact*. Define what it means for a property to be a *topological property*.

(i) Prove that "compactness" is a topological property. If you are using any statements from the course here, prove them as well.

(ii) Prove that a compact subset of a metric space is bounded.

(iii) Let X and Y be compact topological spaces. Assume that Y is Hausdorff and let $f : X \to Y$ be continuous and bijective. Prove that f must be a homeomorphism.

3. Let $B[a, b]$ be the space of all bounded functions on $[a, b]$ and let $C[a, b]$ be the space of all continuous functions on $[a, b]$.

(i) Prove that $B[a, b]$ with sup-metric is complete.

(ii) Let L_K be the space of all functions from $C[a, b]$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [a, b]$. Prove that L_K is closed in $C[a, b]$.

(iii) Let $L = \bigcup_K L_K$ be the space of all functions from $C[a, b]$ which belong to L_K for some K. Prove that L is not closed and its closure in $C[a, b]$ is $C[a, b]$ (you may use other properties of the space $C[a, b]$ in this part without proof).

4. Give the definition of *open* sets and of *closed* sets in a metric space.

(i) Prove that every open subset of $\mathbf{R}$ is a union of pairwise disjoint open intervals.

(ii) Show that the union in (i) is finite or countable.

(iii) Show that the standard (Euclidean) topology of $\mathbf{R}^2$ and its product topology are the same.

5. Let (X, d) be a metric space. We will say that a sequence $\{B_n\}_{n=1}^{\infty}$ of balls is a decreasing shrinking sequence of closed balls if
(1) all B_n are closed;
(2) $B_{n+1} \subset B_n$ for all $n \in \mathbf{N}$;
(3) the sequence of radii of B_n converges to zero.

(i) Prove that if (X, d) is complete, then the intersection of every sequence of decreasing shrinking closed balls is nonempty.

(ii) Prove that if the intersection of every sequence of decreasing shrinking closed balls is nonempty, then (X, d) is complete.