Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M3N9/M4N9

Finite Difference Methods for Partial Differential Equations

Date: Tuesday, 9th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. A three level difference scheme for approximating

$$u_t = u_{xx} \qquad t > 0, \qquad\qquad -\infty < x < \infty,$$
$$u(x,0) = u^0(x),$$

on the uniform grid $(j\Delta x, n\Delta t)$ is

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\,\Delta t} + \frac{(\Delta x)^2}{12}\left[\frac{U_j^{n+1} - 2\,U_j^n + U_j^{n-1}}{(\Delta t)^2}\right] = \frac{U_{j+1}^n - 2\,U_j^n + U_{j-1}^n}{(\Delta x)^2}\,,$$

where U_j^n is the generated approximation to $u(j\Delta x, n\Delta t)$.

(i) Show that the truncation error of this scheme is

$$O(\,(\Delta x)^6) \qquad\qquad \text{if } \Delta t = (\Delta x)^2/(60)^{\frac{1}{2}},$$
$$O(\,(\Delta t)^2 + (\Delta x)^4) \qquad \text{otherwise.}$$

(ii) Use Fourier analysis to find a necessary condition on Δt for the scheme to be stable and have no growth.

You may use the result that the roots z_i of the quadratic $z^2 + p\,z + q = 0$, with $p,\, q \in \mathbb{R}$, satisfy $|z_i| \leq 1$, $i = 1,\, 2$, if and only if $|q| \leq 1$ and $|p| \leq 1 + q$.

2. A numerical solution to the two dimensional heat equation

$$u_t = u_{xx} + u_{yy} \qquad 0 < x,\, y < 1, \quad t > 0,$$

with boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0 \qquad \text{for} \quad 0 \le x,\, y \le 1, \quad t > 0,$$

and initial condition

$$u(x,y,0) = u^0(x,y) \qquad \text{for} \quad 0 \le x,\, y \le 1\,;$$

is to be computed on the uniform mesh $(jh, kh, n\Delta t)$ with $Jh = 1$.

Consider the interior difference schemes

(a) $U^{n+1}_{j,k} = (1 + r\,\delta_x^2 + r\,\delta_y^2\,)\,U^n_{j,k}\,,$

(b) $U^{n+1}_{j,k} = (1 + r\,\delta_x^2)\,(1 + r\,\delta_y^2\,)\,U^n_{j,k}\,,$

$j,\, k = 1 \to J-1, \quad n \ge 0\,;$

where $U^n_{j,k}$ is the generated approximation to $u(jh, kh, n\Delta t)$, $r = \Delta t/h^2$,

$$\delta_x^2\,U^n_{j,k} \equiv U^n_{j-1,k} - 2\,U^n_{j,k} + U^n_{j+1,k} \qquad \text{and} \qquad \delta_y^2\,U^n_{j,k} \equiv U^n_{j,k-1} - 2\,U^n_{j,k} + U^n_{j,k+1}\,.$$

(i) For each scheme find an upper bound on r such that for all $n \ge 0$

$$\max_{0 \le j,\, k \le J} |U^{n+1}_{j,k}| \le \max_{0 \le j,\, k \le J} |U^n_{j,k}|\,,$$

and hence state a sufficient condition for the scheme to be stable. Compare these stability conditions with those obtained using Fourier analysis.

(ii) Write down the corresponding difference schemes for the three dimensional heat equation and obtain stability conditions using Fourier analysis.

3. A numerical solution to the coupled system

$$\begin{aligned} v_t &= -w_{xx} \\ w_t &= v_{xx} \end{aligned} \qquad 0 < x < 1, \quad t > 0,$$

with boundary conditions

$$v(0,t) = v(1,t) = w(0,t) = w(1,t) = 0 \qquad t > 0,$$

and initial conditions

$$v(x,0) = v^0(x), \qquad w(x,0) = w^0(x) \qquad \text{for} \quad 0 \le x \le 1\,;$$

is to be computed on the uniform mesh $(j\Delta x, n\Delta t)$ with $J\Delta x = 1$.

Consider the θ-method

$$\left.\begin{aligned} V_j^{n+1} - V_j^n &= -r\,[\,\theta\,\delta^2\,W_j^{n+1} + (1-\theta)\,\delta^2\,W_j^n\,]\,, \\ W_j^{n+1} - W_j^n &= r\,[\,(1-\theta)\,\delta^2\,V_j^{n+1} + \theta\,\delta^2\,V_j^n\,] \end{aligned}\right. \qquad j = 1 \to J-1, \quad n \ge 0\,;$$

where V_j^n and W_j^n are the generated approximations to $v(j\Delta x, n\Delta t)$ and $w(j\Delta x, n\Delta t)$, $\theta \in [0,1]$, $r = \Delta t/(\Delta x)^2$, and $\delta^2\,V_j^n \equiv V_{j-1}^n - 2\,V_j^n + V_{j+1}^n\,.$

(i) For what values of θ can the scheme be considered explicit?

(ii) Show that the amplification matrix, $G(\Delta t, \Delta x; k)$, for the scheme is

$$\frac{1}{1+\mu^2\,\theta\,(1-\theta)} \begin{pmatrix} 1-\mu^2\,\theta^2 & \mu \\ -\mu & 1-\mu^2\,(1-\theta)^2 \end{pmatrix},$$

where $\mu = 2\,r\,(1-\cos\xi)$ and $\xi = k\,\Delta x$.

(iii) Show that the characteristic polynomial of $G(\Delta t, \Delta x; k)$ is

$$\lambda^2 - b\,\lambda + 1, \qquad \text{where} \qquad b = \frac{2-\mu^2\,[\,\theta^2 + (1-\theta)^2]}{1+\mu^2\,\theta\,(1-\theta)}\,.$$

(iv) Find a necessary condition for the scheme to be stable and have no growth.

(v) Show that $G(\Delta t, \Delta x; k)$ is orthogonal if $\theta = \frac{1}{2}$, and hence in this case show that the scheme is unconditionally stable with no growth.

4. A numerical solution to the two dimensional convection equation

$$\begin{aligned} u_t + a\,u_x + b\,u_y &= 0 \qquad t > 0, \\ u(x,y,0) &= u^0(x,y), \end{aligned} \qquad -\infty < x,\, y < \infty,$$

where $a,\, b \in \mathbb{R}$, is to be approximated on the uniform grid $(jh, kh, n\Delta t)$.

Consider the following schemes

(a) The Crank-Nicolson scheme:

$$(\,1 + \frac{q_1}{2}\,\Delta_x + \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^{n+1} = (\,1 - \frac{q_1}{2}\,\Delta_x - \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^{n}\,,$$

where $q_1 = a\,\Delta t/h$, $q_2 = b\,\Delta t/h$,

$$\Delta_x\,U_{j,k}^n \equiv \frac{1}{2}\,(\,U_{j+1,k}^n - U_{j-1,k}^n\,) \qquad \text{and} \qquad \Delta_y\,U_{j,k}^n \equiv \frac{1}{2}\,(\,U_{j,k+1}^n - U_{j,k-1}^n\,)\,;$$

(b) The ADI scheme:

$$\begin{aligned} (\,1 + \frac{q_1}{2}\,\Delta_x\,)\,U_{j,k}^{n+1,*} &= (\,1 - \frac{q_1}{2}\,\Delta_x - q_2\,\Delta_y\,)\,U_{j,k}^n\,, \\ (\,1 + \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^{n+1} \quad &= (\,1 - \frac{q_1}{2}\,\Delta_x - \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^n - \frac{q_1}{2}\,\Delta_x\,U_{j,k}^{n+1,*}. \end{aligned}$$

Show that the ADI scheme can be rewritten as

$$(\,1 + \frac{q_1}{2}\,\Delta_x)\,(\,1 + \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^{n+1} = (\,1 - \frac{q_1}{2}\,\Delta_x)\,(1 - \frac{q_2}{2}\,\Delta_y\,)\,U_{j,k}^n\,.$$

Hence show that both the schemes (a) and (b) are unconditionally stable and have a truncation error of $O(\,(\Delta t)^2 + h^2)$.

What is the advantage of the ADI scheme over the Crank-Nicolson scheme ?

5. Show that the upwind scheme on the uniform grid $(j\Delta x, n\Delta t)$ for the one dimensional convection equation

$$u_t + a\,u_x = 0 \qquad t > 0, \qquad\qquad -\infty < x < \infty,$$
$$u(x,0) = u^0(x),$$

with $a \in \mathbb{R}$, can be written as

$$U_j^{n+1} = \frac{s}{2}\,(b+a)\,U_{j-1}^n + (1 - s\,b)\,U_j^n + \frac{s}{2}\,(b-a)\,U_{j+1}^n\,,$$

where U_j^n is the generated approximation to $u(j\Delta x, n\Delta t)$, $b = |a|$ and $s = \frac{\Delta t}{\Delta x}$. Find a condition, involving b and s, that is necessary and sufficient for stability.

Let $\underline{u}(x,t) \in \mathbb{R}^p$ satisfy the system

$$\underline{u}_t + A\,\underline{u}_x = \underline{0} \qquad t > 0, \qquad\qquad -\infty < x < \infty,$$
$$\underline{u}(x,0) = \underline{u}^0(x),$$

where $A \in \mathbb{R}^{p\times p}$ has p real eigenvalues with p linearly independent eigenvectors. By diagonalising, generalise the upwind scheme to the system above to obtain

$$\underline{U}_j^{n+1} = \frac{s}{2}\,(B+A)\,\underline{U}_{j-1}^n + (I - s\,B)\,\underline{U}_j^n + \frac{s}{2}\,(B-A)\,\underline{U}_{j+1}^n\,,$$

where $\underline{U}_j^n$ is the generated approximation to $\underline{u}(j\Delta x, n\Delta t)$, $I \in \mathbb{R}^{p\times p}$ is the identity matrix and $B \in \mathbb{R}^{p\times p}$ is a matrix which you are required to define carefully.

Under what circumstances is $B \pm A = 0$?

What is a necessary and sufficient condition for stability of this scheme ?