Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2007

This paper is also taken for the relevant examination for the Associateship.

M3N8/M4N8 FINITE ELEMENT METHOD

Date: Tuesday, 22th May 2007 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Let Ω be a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega \equiv \partial_1\Omega \cup \partial_2\Omega$, either subset being possibly empty. Given $\sigma \in C^1(\overline{\Omega})$ with $\sigma(x,y) \geq \sigma_0 > 0$ for all $(x,y) \in \overline{\Omega}$, $c,\ \alpha,\ g_1,\ g_2 \in \mathbb{R}$, with $c,\ \alpha \geq 0$, and $f \in L^2(\Omega)$; consider the following problem:

(P1) Find u such that

$$-\nabla\,.\,(\sigma\,\nabla u) + c\,u = f \qquad \text{in } \Omega, \qquad u = g_1 \quad \text{on } \partial_1\Omega, \qquad \sigma\,\frac{\partial u}{\partial n} + \alpha\,u = g_2 \quad \text{on } \partial_2\Omega;$$

where $\dfrac{\partial u}{\partial n} \equiv \underline{\nabla} u\,.\,\underline{n}$ and $\underline{n}$ is the outward unit normal to $\partial\Omega$.

For any $g \in \mathbb{R}$, let $V(g) := \{v \in H^1(\Omega) : v = g \text{ on } \partial_1\Omega\}$. Show that a solution of (P1) is a solution of the following problem:

(P2) Find $u \in V(g_1)$ such that

$$a(u,v) = \int_\Omega f\,v\,\mathrm{d}x\,\mathrm{d}y \;+\; g_2 \int_{\partial_2\Omega} v\,\mathrm{d}s \qquad \forall\ v \in V(0)\,;$$

where

$$a(w,v) := \int_\Omega [\,\sigma\,\underline{\nabla} w\,.\,\underline{\nabla} v + c\,w\,v\,]\,\mathrm{d}x\,\mathrm{d}y \;+\; \alpha \int_{\partial_2\Omega} w\,v\,\mathrm{d}s \qquad \forall\ w,\ v \in H^1(\Omega).$$

Show that problem (P2) is *equivalent* to the following problem:

(P3) Find $u \in V(g_1)$ such that

$$J(u) \leq J(w) := a(w,w)\; - 2\int_\Omega f\,w\,\mathrm{d}x\,\mathrm{d}y \;-\; 2\,g_2 \int_{\partial_2\Omega} w\,\mathrm{d}s \qquad \forall\ w \in V(g_1).$$

State, without proof, conditions on the data $c,\ \alpha,\ g_1,\ g_2,\ f$ and $\partial_1\Omega$ to guarantee the existence and uniqueness of a solution to (P2).

Give an example of a choice of data for which

(i) There exists a non-unique solution to (P2).

(ii) There does not exist a solution to (P2).

2. Let $\Omega \equiv (a, b)$ be a bounded interval. For all $f \in L^2(\Omega)$ and $g \in \mathbb{R}$, assume there exists a solution $w \in H^2(\Omega)$, dependent on f and g, to the problem

$$-w'' + w = f \qquad \text{in } \Omega, \qquad\qquad w(a) = 0, \qquad w'(b) + w(b) = g.$$

By considering its weak formulation show that for any given f and g, the solution w is unique and that

$$\|w\|_{1,\Omega} \leq \|f\|_{0,\Omega} + |g|\,,$$

where for any $m \in \mathbb{N}$

$$\|v\|_{m,\Omega} := \left\{ \sum_{i=0}^{m} \int_a^b [v^{(i)}]^2 \,\mathrm{d}x \right\}^{\frac{1}{2}}$$

is the norm on the Sobolev space $H^m(\Omega)$.

Hence deduce from the differential equation that for any $r \in \mathbb{N}$, if $f \in H^r(\Omega)$ then there exists a positive constant M, dependent only on r, such that

$$\|w\|_{r+2,\Omega} \leq M\,[\,\|f\|_{r,\Omega} + |g|\,].$$

Let $a = x_0 < x_1 \cdots < x_{j-1} < x_j \cdots < x_{J-1} < x_J = b$, $h_j := x_j - x_{j-1}$, $j = 1 \to J$, and $h := \max_{j=1\to J} h_j$. For any positive integer k, let

$$V_k^h := \left\{ v^h \in C[a, b] : v^h \text{ is a polynomial of degree } k \text{ or less on } [x_{j-1}, x_j],\ j = 1 \to J \right\}.$$

Formulate the finite element approximation $w_k^h \in V_k^h$ to the above problem. Show that for any given $f \in L^2(\Omega)$ and $g \in \mathbb{R}$, w_k^h exists and is unique.

Setting $e := w - w_k^h$, show that there exists a positive constant C_1 such that

$$\|e\|_{1,\Omega} \leq C_1\, h^k\,[\,\|f\|_{k-1,\Omega} + |g|\,].$$

By considering the weak formulation of the auxiliary problem: find z such that

$$-z'' + z = e \qquad \text{in } \Omega, \qquad\qquad z(a) = z'(b) + z(b) = 0;$$

show that there exists a positive constant C_2 such that

$$\|e\|_{0,\Omega} \leq C_2\, h^{k+1}\,[\,\|f\|_{k-1,\Omega} + |g|\,].$$

$\Big[$ *You may assume the approximation result that there exists a positive constant C, independent of h, such that*

$$\|v - v_{k,I}^h\|_{1,\Omega} \leq C\, h^k\, \|v\|_{k+1,\Omega} \qquad \forall\, v \in H^{k+1}(\Omega),$$

where $v_{k,I}^h \in V_k^h$ is the interpolant of v, such that for $j = 1 \to J$

$$v_{k,I}^h(x_{j-\frac{s}{k}}) = v(x_{j-\frac{s}{k}}), \quad s = 0,\, 1, \cdots, k; \qquad \text{where} \quad x_{j-\frac{s}{k}} := (1 - \tfrac{s}{k})\, x_j + \tfrac{s}{k}\, x_{j-1}\,.\Big]$$

3. Let $\widehat{e}$ be the tetrahedron in $(\widehat{x}, \widehat{y}, \widehat{z})$ space with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively.

For $i \neq j$, let

$$\widehat{P}_{ij} := \tfrac{1}{2}\,[\widehat{P}_i + \widehat{P}_j].$$

Consider the following quadrature rules

$$\begin{aligned}
\widehat{Q}^a_{\widehat{e}}(\widehat{v}) \;&:=\; \tfrac{1}{24}\left[\widehat{v}(\widehat{P}_1) + \widehat{v}(\widehat{P}_2) + \widehat{v}(\widehat{P}_3) + \widehat{v}(\widehat{P}_4)\right],\\
\widehat{Q}^b_{\widehat{e}}(\widehat{v}) \;&:=\; \tfrac{1}{36}\left[\widehat{v}(\widehat{P}_{12}) + \widehat{v}(\widehat{P}_{13}) + \widehat{v}(\widehat{P}_{14}) + \widehat{v}(\widehat{P}_{23}) + \widehat{v}(\widehat{P}_{24}) + \widehat{v}(\widehat{P}_{34})\right],
\end{aligned}$$

approximating

$$\int_{\widehat{e}} \widehat{v}(\widehat{x}, \widehat{y}, \widehat{z})\, \mathrm{d}\widehat{x}\, \mathrm{d}\widehat{y}\, \mathrm{d}\widehat{z}.$$

Let

$$\mathcal{P}_k(\widehat{x}, \widehat{y}, \widehat{z}) := \{\,\text{all polynomials in } \widehat{x}, \widehat{y} \text{ and } \widehat{z} \text{ of degree } \leq k\,\}.$$

Show that $\widehat{Q}^a_{\widehat{e}}(\widehat{v})$ and $\widehat{Q}^b_{\widehat{e}}(\widehat{v})$ are exact for all $\widehat{v} \in \mathcal{P}_1(\widehat{x}, \widehat{y}, \widehat{z})$, but not for all $\widehat{v} \in \mathcal{P}_2(\widehat{x}, \widehat{y}, \widehat{z})$.

[*You may use the result that*

$$\int_{\widehat{e}} \widehat{x}^i\, \widehat{y}^j\, \widehat{z}^k\, \mathrm{d}\widehat{x}\, \mathrm{d}\widehat{y}\, \mathrm{d}\widehat{z} = \frac{i!\; j!\; k!}{(i+j+k+3)!} \quad \forall\, i,\, j,\, k \in \mathbb{N}.\,]$$

Find ω such that

$$\omega\, \widehat{Q}^a_{\widehat{e}}(\widehat{v}) + (1-\omega)\, \widehat{Q}^b_{\widehat{e}}(\widehat{v})$$

is exact for all $\widehat{v} \in \mathcal{P}_2(\widehat{x}, \widehat{y}, \widehat{z})$.

Show that this quadrature rule is not exact for all $\widehat{v} \in \mathcal{P}_3(\widehat{x}, \widehat{y}, \widehat{z})$.

Let e be the tetrahedron with vertices P_i, having coordinates (x_i, y_i, z_i), $i = 1 \to 4$. Derive a quadrature rule approximating

$$\int_e v(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

which is exact for all $v \in \mathcal{P}_2(x, y, z)$. State precisely the sampling points in terms of the coordinates (x_i, y_i, z_i), $i = 1 \to 4$, and the weights in terms of the volume of e.

4. Let τ be a triangle with vertices P_1, P_2 and P_3. For $i = 1 \to 3$, let $\phi_i(x, y)$ be the linear function such that

$$\phi_i(P_j) \;=\; \delta_{i,j} \quad j = 1 \to 3.$$

State, without proof, the entries

$$\int_\tau \underline{\nabla}\phi_i \,.\, \underline{\nabla}\phi_j \,\mathrm{d}x\,\mathrm{d}y \qquad\qquad i, j = 1 \to 3$$

of the "element stiffness matrix" for τ in terms of the cotangents of its angles.

Consider the problem: Find u such that

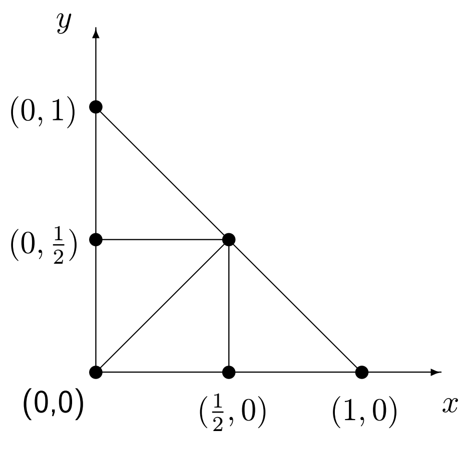
$$\nabla^2 u \;=\; 6 \qquad \text{in the triangle} \quad 0 < y < 1 - x, \quad 0 < x < 1;$$

subject to the boundary conditions

$$u(x, 1-x) = 3\,x^2 + 2\,x + 3, \qquad \frac{\partial u}{\partial y}(x, 0) = 0 \qquad \text{for} \quad 0 \le x \le 1;$$

$$\text{and} \qquad \frac{\partial u}{\partial x}(0, y) = 4 \qquad \text{for} \quad 0 \le y \le 1.$$

Formulate and compute the continuous piecewise linear approximation to the above problem based on the triangulation given in the figure below, where the nodes $1 \to 6$ have (x, y) coordinates $(0,0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$ respectively.



5. Let $\widehat{e}$ be the square in the $(\widehat{x},\widehat{y})$ plane with vertices $(-1,-1)$, $(1,-1)$, $(-1,1)$ and $(1,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. In addition there are nodes $\widehat{P}_5$, $\widehat{P}_6$, $\widehat{P}_7$ and $\widehat{P}_8$ on $\widehat{e}$ with coordinates $(0,-1)$, $(-1,0)$, $(1,0)$ and $(0,1)$ respectively. Let B be the set of functions defined on $\widehat{e}$ such that

$$f \in B \Longrightarrow f(\widehat{x},\widehat{y}) = a_1 + a_2\,\widehat{x} + a_3\,\widehat{y} + a_4\,\widehat{x}^2 + a_5\,\widehat{x}\,\widehat{y} + a_6\,\widehat{y}^2 + a_7\,\widehat{x}^2\,\widehat{y} + a_8\,\widehat{x}\,\widehat{y}^2$$

for some constants $\{a_i\}_{i=1}^8$. Let $\{\widehat{\phi}_i\}_{i=1}^8$ be the basis functions such that

$$\widehat{\phi}_i \in B \qquad \text{and} \qquad \widehat{\phi}_i(\widehat{P}_j) \;=\; \delta_{i,j} \qquad i,\, j = 1 \to 8.$$

Find $\widehat{\phi}_1$.

Let the points P_j have coordinates (x_j, y_j), $j = 1 \to 8$, such that $P_j \equiv \widehat{P}_j$ for $j = 2 \to 8$; $x_1 < 0$ and $y_1 < 0$. Consider the mapping $F\,:(\widehat{x},\widehat{y}) \in \widehat{e} \to (x,y)$ given by

$$x \;=\; \sum_{i=1}^{8} x_i\,\widehat{\phi}_i(\widehat{x},\widehat{y}) \quad \text{and} \quad y \;=\; \sum_{i=1}^{8} y_i\,\widehat{\phi}_i(\widehat{x},\widehat{y}).$$

Sketch the image, e, of $\widehat{e}$ under the map F.

Show that for $(\widehat{x},\widehat{y}) \in \widehat{e}$

$$\tfrac{9}{16} \geq \frac{\partial\widehat{\phi}_1}{\partial\widehat{x}},\ \frac{\partial\widehat{\phi}_1}{\partial\widehat{y}} \geq -\tfrac{3}{2}\,;$$

and use this result to show that

$$\tfrac{2}{3} > (X+Y) > -\tfrac{16}{9}, \qquad 1 - \tfrac{3}{2}\,X + \tfrac{9}{16}\,Y > 0, \qquad 1 + \tfrac{9}{16}\,X - \tfrac{3}{2}\,Y > 0,$$

where $X = x_1 + 1$ and $Y = y_1 + 1$, are sufficient conditions for F to be invertible.

Assuming that the above conditions hold; find $\underline{\nabla}\phi_1(1,-1)$, where

$$\phi_1(x,y) := \widehat{\phi}_1(F^{-1}(x,y)) \qquad \forall\ (x,y) \in e.$$