Imperial College London

UNIVERSITY OF LONDON

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Setter:	Barrett
Checker:	G. Moore
Editor:	G. Moore
External:	Cowley
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M3N8/M4N8/MSA8 FINITE ELEMENT METHOD

Date: ? 2005 Time: ?

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Let Ω be the square in the x-y plane with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Given $f \in L^2(\Omega)$ and g_1, $g_2 \in C^1[0,1]$ such that

$$g_i(0) = g_i(1) \qquad \text{and} \qquad g_i'(0) = g_i'(1) \qquad i = 1,\, 2;$$

consider the following problem :

(P1) Find $u(x,y)$ such that

$$\begin{array}{llll} -\nabla^2 u = f & \text{in } \Omega, & & \\ u(x,0) = u(x,1) & \text{and} \quad u_y(x,0) = u_y(x,1) & \text{for} & x \in (0,1), \\ u(0,y) = g_1(y) & \text{and} \quad u(1,y) \;= g_2(y) & \text{for} & y \in [0,1]. \end{array}$$

Let

$$V := \{\, v \in H^1(\Omega) \;:\; v(x,0) = v(x,1) \quad \forall\, x \in (0,1)\,\}$$

and $$V(g_1,g_2) := \{\, v \in V \;:\; v(0,y) = g_1(y), \quad v(1,y) = g_2(y) \quad \forall\, y \in [0,1]\,\}.$$

Show that a solution of (P1) is a solution of the following problem :

(P2) Find $u \in V(g_1,g_2)$ such that

$$\int_\Omega \underline{\nabla} u \cdot \underline{\nabla} v \, dx\, dy = \int_\Omega f\, v \, dx\, dy \qquad \forall\, v \in V(0,0).$$

Show that the solution of (P1) is unique.

Show that problem (P2) is *equivalent* to the following problem :

(P3) Find $u \in V(g_1,g_2)$ such that

$$J(u) \le J(w) \qquad \forall\, w \in V(g_1,g_2),$$

where
$$J(w) := \tfrac{1}{2} \int_\Omega |\underline{\nabla} w|^2 \, dx\, dy - \int_\Omega f\, w \, dx\, dy\,.$$

2. Given constants $a > 0$ and $b \geq 0$, let A be the differential operator

$$A\,v := a\,\frac{d^4v}{dx^4} - b\,\frac{d^2v}{dx^2} + v.$$

For all $f \in L^2(\Omega)$, where $\Omega \equiv (0,1) \subset \mathbb{R}$, assume there exists a solution $w \in H^4(\Omega)$, dependent on f, to the problem

$$A\,w = f \qquad \text{in } \Omega, \qquad\qquad w(0) = \frac{dw}{dx}(0) = w(1) = \frac{dw}{dx}(1) = 0.$$

By considering its weak formulation, show that for any given $f \in L^2(\Omega)$ the solution w is unique and that

$$a\,|w|^2_{2,\Omega} + b\,|w|^2_{1,\Omega} + |w|^2_{0,\Omega} \leq |f|^2_{0,\Omega},$$

where for any $m \in \mathbb{N}$

$$|v|_{m,\Omega} := \left\{ \int_\Omega \left[\frac{d^m v}{dx^m}\right]^2 \mathrm{d}x \right\}^{\frac{1}{2}}.$$

Hence deduce from the differential equation that

$$|w|_{4,\Omega} \leq C_1\,|f|_{0,\Omega},$$

where throughout this question, C_i denote positive constants, possibly dependent on a and b.

Let

$$S^h := \{\, v^h \in C^1(\overline{\Omega}) : v^h \text{ cubic on } [x_{j-1}, x_j], \quad j = 1 \to J\,\},$$

where $0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = 1$. Let $h := \max_{j=1\to J}(x_j - x_{j-1})$. Formulate the S^h finite element approximation to the above problem. Show that for any given $f \in L^2(\Omega)$, the resulting finite element approximation, w^h, exists and is unique.

Setting $e := w - w^h$, show that there exists a C_2, independent of h, such that

$$a\,|e|^2_{2,\Omega} + b\,|e|^2_{1,\Omega} + |e|^2_{0,\Omega} \leq C_2\,h^4\,|f|^2_{0,\Omega}.$$

$\Big[$ *You may assume the approximation result that there exists a C_3, independent of h, such that for $m = 0$, 1 and 2*

$$|v - v^h_I|_{m,\Omega} \leq C_3\,h^{4-m}\,|v|_{4,\Omega} \qquad \forall\, v \in H^4(\Omega),$$

where $v^h_I \in S^h$ and $v^h_I(x_j) = v(x_j)$, $\dfrac{dv^h_I}{dx}(x_j) = \dfrac{dv}{dx}(x_j) \quad j = 0 \to J$ $\Big]$.

Question 2 continued over $\cdots$

Write down the weak formulation of the auxiliary problem: Find z such that

$$A\,z = e \qquad \text{in } \Omega, \qquad\qquad z(0) = \frac{dz}{dx}(0) = z(1) = \frac{dz}{dx}(1) = 0.$$

Show that

$$|e|^2_{0,\Omega} = \int_\Omega \left[a\,\frac{d^2 e}{dx^2}\,\frac{d^2}{dx^2}(z - z_I^h) + b\,\frac{de}{dx}\,\frac{d}{dx}(z - z_I^h) + e\,(z - z_I^h) \right] \mathrm{d}x$$

and hence that there exists a C_4, independent of h, such that

$$|e|_{m,\Omega} \le C_4\, h^{4-m}\, |f|_{0,\Omega}, \qquad\qquad m = 0,\ 1 \text{ and } 2.$$

3. For $\alpha \in [0,1]$, consider the following approximation to $\displaystyle\int_{-1}^{1} u(\xi)\,d\xi$:

$$Q_\alpha(u) \equiv u(-\alpha) + u(\alpha).$$

Show that $Q_\alpha(u)$ is exact for all linear u for any $\alpha \in [0,1]$.

Find the unique $\alpha^\star$ so that $Q_{\alpha^\star}(u)$ is exact for all cubic u.

Let e be the square $[-1,1] \times [-1,1]$ in the ξ–η plane. Using $Q_{\alpha^\star}$, deduce a quadrature rule consisting of 4 sampling points which approximates

$$\int_e v(\xi,\eta)\,d\xi\,d\eta$$

and is exact for all bicubic v.

Consider the mapping $G : (\xi,\eta) \to (x,y)$ given by

$$x = \tfrac{1}{2}(1+\xi) \qquad\qquad y = \tfrac{1}{4}(1-\xi)(1+\eta).$$

Find the image of e under the map G.

Let τ be the triangle in the x–y plane with vertices (0,0), (1,0) and (0,1). Using the results above deduce a quadrature rule with 4 sampling points in the interior of τ which approximates

$$\int_\tau w(x,y)\,dx\,dy$$

and is exact for all quadratic w. [State precisely the sampling points and weights in terms of $\beta := \frac{1}{2}(1+\alpha^\star)$.]

4. Let $\widehat{e}$ be the square with vertices $(-1,-1),(1,-1),(-1,1)$ and $(1,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. For $i = 1 \to 4$, find the bilinear basis functions $\widehat{\phi}_i(\widehat{x},\widehat{y}),\ (\widehat{x},\widehat{y}) \in \widehat{e}$, such that

$$\widehat{\phi}_i(\widehat{P}_j) \ = \ \delta_{ij} \quad j = 1 \to 4.$$

For i, $j = 1 \to 4$, show that

$$\int_{\widehat{e}} \frac{\partial \widehat{\phi}_i}{\partial \widehat{x}} \frac{\partial \widehat{\phi}_j}{\partial \widehat{x}} \, \mathrm{d}\widehat{x} \, \mathrm{d}\widehat{y} \ = \begin{cases} \ \ \frac{1}{3} & \text{if } i = j \\ \ \ \frac{1}{6} & \text{if } |i-j| = 2 \\ -\frac{1}{6} & \text{if } i + j = 5 \\ -\frac{1}{3} & \text{otherwise} \end{cases}$$

and hence

$$\int_{\widehat{e}} \widehat{\underline{\nabla}} \widehat{\phi}_i \,.\, \widehat{\underline{\nabla}} \widehat{\phi}_j \, \mathrm{d}\widehat{x} \, \mathrm{d}\widehat{y} = \begin{cases} \ \ \frac{2}{3} & \text{if } i = j \\ -\frac{1}{3} & \text{if } i + j = 5 \\ -\frac{1}{6} & \text{otherwise} \end{cases} ,$$

where $\widehat{\underline{\nabla}} \widehat{\phi}_i := (\dfrac{\partial \widehat{\phi}_i}{\partial \widehat{x}}, \dfrac{\partial \widehat{\phi}_i}{\partial \widehat{y}})^T$.

Let e be the square with vertices (a,b), $(a+h,b)$, $(a,b+h)$ and $(a+h,b+h)$, where $h > 0$. For $i = 1 \to 4$ define

$$\phi_i(x,y) = \widehat{\phi}_i \left(\tfrac{2(x-a)-h}{h}, \tfrac{2(y-b)-h}{h} \right) \quad \forall \ (x,y) \in e.$$

For i, $j = 1 \to 4$, show that the integrals

$$\int_e \underline{\nabla} \phi_i \,.\, \underline{\nabla} \phi_j \, \mathrm{d}x \, \mathrm{d}y = \int_{\widehat{e}} \widehat{\underline{\nabla}} \widehat{\phi}_i \,.\, \widehat{\underline{\nabla}} \widehat{\phi}_j \, \mathrm{d}\widehat{x} \, \mathrm{d}\widehat{y} \,.$$

Consider the problem: Find u such that

$$\nabla^2 u = 0 \qquad \text{on the square} \qquad 0 < x < 2, \quad 0 < y < 2;$$

subject to the boundary conditions

$$\begin{aligned} u(x,0) &= x\,(x-4), & u(x,2) \ &= (x-2)^2 & \text{for } x \in (0,2); \\ u(0,y) &= y\,(4-y), & \frac{\partial u}{\partial x}(2,y) &= 0 & \text{for } y \in (0,2). \end{aligned}$$

Formulate and compute the continuous piecewise bilinear approximation to this problem on uniform squares with sides of unit length.

5. Let $\widehat{e}$ be the triangle in the $(\widehat{x}, \widehat{y})$ plane with vertices $(0,0)$, $(1,0)$ and $(0,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$ and $\widehat{P}_3$ respectively. In addition there are nodes $\widehat{P}_4$, $\widehat{P}_5$ and $\widehat{P}_6$ on $\widehat{e}$ with coordinates $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ respectively.

For $i = 1 \to 6$, find the quadratic basis functions $\widehat{\phi}_i(\widehat{x}, \widehat{y})$, $(\widehat{x}, \widehat{y}) \in \widehat{e}$, such that

$$\widehat{\phi}_i(\widehat{P}_j) \ = \ \delta_{ij} \qquad j = 1 \to 6.$$

Given points P_i with coordinates (x_i, y_i), $i = 1 \to 6$, such that $P_j \equiv \widehat{P}_j$ for $j = 1 \to 5$; consider the mapping $F : (\widehat{x}, \widehat{y}) \in \widehat{e} \to (x, y)$ defined by

$$x \ = \ \sum_{i=1}^{6} x_i \, \widehat{\phi}_i(\widehat{x}, \widehat{y}) \quad \text{and} \quad y \ = \ \sum_{i=1}^{6} y_i \, \widehat{\phi}_i(\widehat{x}, \widehat{y}).$$

Find sufficient conditions on x_6 and y_6 to ensure that F is invertible. Hence find the area of e, where e is the image of $\widehat{e}$ under the map F.

Assuming that F is invertible; define

$$\phi_i(x, y) := \widehat{\phi}_i(F^{-1}(x, y)) \quad \forall \, (x, y) \in e, \quad i = 1 \to 6.$$

Find

(i) $\phi_j(x_c, y_c)$, $j = 1 \to 6$, where $x_c := (1 + 4\,x_6)/9$ and $y_c := (1 + 4\,y_6)/9$.

(ii) $\underline{\nabla}\phi_6(x_6, y_6)$.