Imperial College London

UNIVERSITY OF LONDON

Course:	M3N8/M4N8/MSA8
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M3N8/M4N8/MSA8 FINITE ELEMENT METHOD

Date: Monday, 24th May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. Given $\alpha,\ g \in \mathbb{R},\ \sigma \in C^2[0,1],\ b \in C^1[0,1]$ and $c,\ f \in C[0,1]$; where

$$\sigma(x) \geq \sigma_0 > 0 \qquad \text{and} \qquad b(x),\ c(x),\ \alpha \geq 0 \qquad \forall \quad x \in [0,1];$$

consider the following problem :

(P1) Find u such that

$$(\sigma\, u'')'' - (b\, u')' + c\, u = f \qquad \text{in } (0,1)$$

$$u''(0) = u''(1) = [\,(\sigma\, u'')' - b\, u'\,](1) = 0 \qquad \text{and} \qquad [\,(\sigma\, u'')' - b\, u' + \alpha\, u\,](0) = g.$$

Show that a solution of (P1) is a solution of

(P2) Find $u \in H^2(0,1)$ such that

$$a(u,v) \;=\; \int_0^1 f\, v\, dx + g\, v(0) \qquad \forall\ v \in H^2(0,1)\,;$$

where

$$a(w,v) := \int_0^1 [\,\sigma w''\, v'' + b\, w'\, v' + c\, w\, v\,]\, dx + \alpha\, w(0)\, v(0) \qquad \forall\ w,\ v \in H^2(0,1).$$

Show that a solution of (P2) is a solution of

(P3) Find $u \in H^2(0,1)$ such that

$$a(u,u) - 2\int_0^1 f\, u\, dx - 2\, g\, u(0) \leq a(v,v) - 2\int_0^1 f\, v\, dx \;- 2\, g\, v(0) \qquad \forall\ v \in H^2(0,1).$$

State conditions on b, c and α in order to ensure that the solution of (P2), and hence (P1), is unique.

Give an example of a choice of data $\sigma,\ b,\ c,\ f,\ g$ and α for which

(i) The solution of (P1) is not unique.

(ii) There does not exist a solution to (P1).

2. For $(a,b) \subset \mathbb{R}$ and $\ell,\, m \in \mathbb{N}$, let

$$|w|_{\ell,(a,b)} := \left[\int_a^b \left(\frac{d^\ell w}{dx^\ell}\right)^2 dx\right]^{\frac{1}{2}}, \qquad \|w\|_{m,(a,b)} := \left[\sum_{\ell=0}^{m} |w|^2_{\ell,(a,b)}\right]^{\frac{1}{2}}$$

$$\text{and} \qquad H^m(a,b) := \{\, w : \|w\|_{m,(a,b)} < \infty \,\}.$$

Let $a = x_0 < x_1 \cdots < x_{j-1} < x_j \cdots < x_{J-1} < x_J = b$, $h_j := x_j - x_{j-1}$, $j = 1 \to J$, and $h := \max_{j=1\to J} h_j$. For any positive integer k, let

$$V_k^h := \left\{ v^h \in C[a,b] : v^h \text{ is a polynomial of degree } k \text{ on } (x_{j-1}, x_j),\ j = 1 \to J \right\}.$$

For any $w \in C[a,b]$, let $w^h_{k,I} \in V_k^h$ be the interpolant of w, such that for $j = 1 \to J$

$$w^h_{k,I}(x_{j-\frac{r}{k}}) = w(x_{j-\frac{r}{k}}), \quad r = 0,\, 1, \cdots, k; \qquad \text{where} \quad x_{j-\frac{r}{k}} := (1 - \tfrac{r}{k})\, x_j + \tfrac{r}{k}\, x_{j-1}\,.$$

Assuming that for all positive integers m, and distinct points $\{y_i\}_{i=1}^m$, $y_i \in [0,1]$, there exists a positive constant $C_1(m)$, such that

$$\|w\|^2_{m,(0,1)} \le C_1(m) \left[|w|^2_{m,(0,1)} + \sum_{i=1}^{m} [w(y_i)]^2 \right] \qquad \forall\, w \in H^m(0,1);$$

show for $\ell = 0$ and 1 that

$$|w - w^h_{k,I}|^2_{\ell,(x_{j-1},x_j)} \le C_1(k+1)\, h_j^{2(k+1-\ell)}\, |w|^2_{k+1,(x_{j-1},x_j)}, \qquad j = 1 \to J.$$

For all $f \in H^r(a,b)$, $r \ge 0$, assume that there exists a unique solution $u \in H^{r+2}(a,b)$, dependent on f, to the problem

$$-\frac{d^2u}{dx^2} + u = f \qquad \text{in } (a,b), \qquad\qquad u(a) = u(b) = 0\,;$$

and that there exists a positive constant $C_2(r)$ such that

$$\|u\|_{r+2,(a,b)} \le C_2(r)\, \|f\|_{r,(a,b)}\,.$$

Formulate the V_k^h finite element approximation, u_k^h, to the above problem.

Assuming that $f \in H^{k-1}(a,b)$; show that there exists a positive constant C_3 such that

$$\|u - u_k^h\|_{1,(a,b)} \le C_3\, h^k\, \|f\|_{k-1,(a,b)}\,.$$

3. Let τ be an *equilateral* triangle with vertices P_1, P_2 and P_3. For $i = 1 \to 3$, let $\phi_i(x, y)$ be the linear function on τ such that

$$\phi_i(P_j) = \delta_{ij} \quad j = 1 \to 3.$$

Show that

$$\int_\tau \phi_i \, dx \, dy = \tfrac{1}{3} \, \underline{m}(\tau) \qquad \text{and} \qquad \int_\tau \underline{\nabla}\phi_i \, . \, \underline{\nabla}\phi_j \, dx \, dy = \begin{cases} \frac{1}{\sqrt{3}} & \text{if } j = i \\ \frac{-1}{2\sqrt{3}} & \text{if } j \neq i \end{cases} .$$

Let Ω be the regular hexagon centred on the origin with vertices A, B, C, D, E and F; see the figure below. Let $\partial_1\Omega$ be that part of the boundary of Ω consisting of the sides AB, BC, EF and FA; and $\partial_2\Omega$ be the remaining sides CD and DE. Consider the problem :

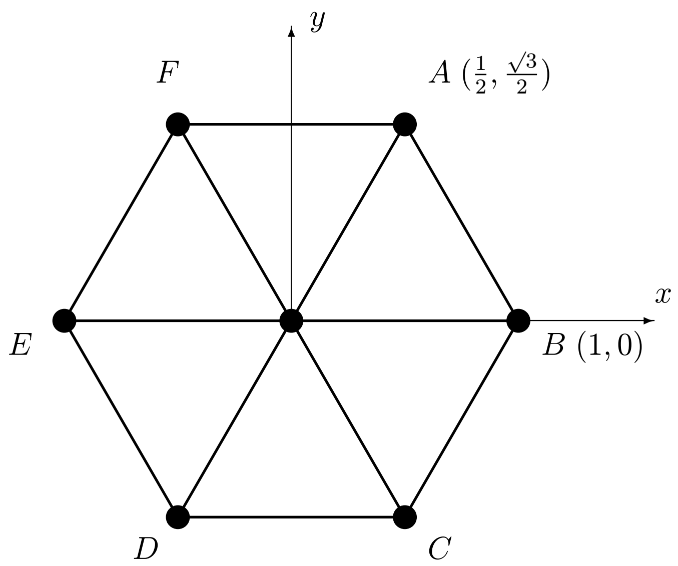
Find u such that

$$-\nabla^2 u = 2 \quad \text{in } \Omega$$

$$u = 2\,(y^2 - x^2) \quad \text{on } \partial_1\Omega \qquad \text{and} \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial_2\Omega,$$

where $\dfrac{\partial u}{\partial n} := \underline{\nabla} u \, . \, \underline{n}$ and $\underline{n}$ is the outward unit normal to $\partial\Omega$.

Formulate and compute the continuous piecewise linear approximation to the above problem based on the equilateral triangulation of Ω given in the figure below.



4. Show that the quadrature rules

$$\widehat{Q}^a_\alpha(\widehat{v}) := \widehat{v}(-\alpha,0) + \widehat{v}(\alpha,0) + \widehat{v}(0,-\alpha) + \widehat{v}(0,\alpha)$$

and
$$\widehat{Q}^b_\beta(\widehat{v}) := \widehat{v}(-\beta,-\beta) + \widehat{v}(-\beta,\beta) + \widehat{v}(\beta,-\beta) + \widehat{v}(\beta,\beta)$$

approximating

$$\int_{\widehat{e}} \widehat{v}(\widehat{x},\widehat{y})\, d\widehat{x}\, d\widehat{y}\,, \quad \text{where} \quad \widehat{e} := [-1,1] \times [-1,1]\,,$$

are exact for all bilinear $\widehat{v}$ for any $\alpha,\ \beta \in [0,1]$.

Find the unique $\alpha^\star \in [0,1]$ so that $\widehat{Q}^a_{\alpha^\star}(\widehat{v})$ is exact for all cubic $\widehat{v}$, and show that $\widehat{Q}^a_{\alpha^\star}(\widehat{v})$ is not exact for all biquadratic $\widehat{v}$.

Find the unique $\beta^\star \in [0,1]$ so that $\widehat{Q}^b_{\beta^\star}(\widehat{v})$ is exact for all bicubic $\widehat{v}$, and show that $\widehat{Q}^b_{\beta^\star}(\widehat{v})$ is not exact for all quartic $\widehat{v}$.

Let e be the parallelogram with vertices (x_i, y_i), $i = 1 \to 4$, such that $x_4 = x_2 + x_3 - x_1$ and $y_4 = y_2 + y_3 - y_1$. Show that the linear map $(x,y)^T = B\,(\widehat{x},\widehat{y})^T + \underline{b}$, where

$$B := \tfrac{1}{2} \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \qquad \text{and} \qquad \underline{b} := \tfrac{1}{2} \begin{pmatrix} x_2 + x_3 \\ y_2 + y_3 \end{pmatrix}\,,$$

transforms $(\widehat{x},\widehat{y}) \in \widehat{e}$ to $(x,y) \in e$.

Use the quadrature rule $\widehat{Q}^b_{\beta^\star}(\cdot)$ over $\widehat{e}$ to develop the corresponding approximation to

$$\int_e v(x,y)\, dx\, dy\,,$$

stating precisely the sampling points and the weights in terms of B and $\underline{b}$.

For what class of functions v is this approximation exact ? Give reasons for your answer.

5. Let $\widehat{e}$ be the square in the $(\widehat{x}, \widehat{y})$ plane with vertices $(-1,-1)$, $(1,-1)$, $(-1,1)$ and $(1,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. In addition there are nodes $\widehat{P}_5$, $\widehat{P}_6$, $\widehat{P}_7$, $\widehat{P}_8$ and $\widehat{P}_9$ on $\widehat{e}$ with coordinates $(0,-1)$, $(-1,0)$, $(0,0)$, $(1,0)$ and $(0,1)$ respectively.

Find $\widehat{\phi}_1(\widehat{x}, \widehat{y})$; where for $i = 1 \to 9$, $\widehat{\phi}_i(\widehat{x}, \widehat{y}),\ (\widehat{x}, \widehat{y}) \in \widehat{e}$, is the *biquadratic* function such that

$$\widehat{\phi}_i(\widehat{P}_j) = \delta_{ij} \qquad j = 1 \to 9.$$

Let the points P_j have coordinates $(x_j, y_j),\ j = 1 \to 9$, such that $P_j \equiv \widehat{P}_j$ for $j \neq 1$ and $x_1,\, y_1 < 0$. Consider the mapping $F\, : (\widehat{x}, \widehat{y}) \in \widehat{e} \to (x, y)$ given by

$$x = \sum_{i=1}^{9} x_i\, \widehat{\phi}_i(\widehat{x}, \widehat{y}) \qquad \text{and} \qquad y = \sum_{i=1}^{9} y_i\, \widehat{\phi}_i(\widehat{x}, \widehat{y}).$$

Sketch the image, e, of $\widehat{e}$ under the map F.

Show that for $(\widehat{x}, \widehat{y}) \in \widehat{e}$

$$0.5 \geq \frac{\partial \widehat{\phi}_1}{\partial \widehat{x}},\ \frac{\partial \widehat{\phi}_1}{\partial \widehat{y}} \geq -1.5\,,$$

and use this result to show that

$$-4 > 3\,(x_1 + y_1) > -12 \qquad \text{and} \qquad y_1 > 3\,x_1 > 9\,y_1$$

are sufficient conditions for F to be invertible.

Assuming that the above conditions hold; find $\underline{\nabla}\phi_1(-1,0)$, where

$$\phi_i(x,y) := \widehat{\phi}_i(F^{-1}(x,y)) \qquad \forall\ (x,y) \in e, \qquad i = 1 \to 9.$$