UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION(MATHEMATICS) MAY-JUNE 2003

This paper is also taken for the relevant examination for the Associateship

M3N8/M4N8/MSA8 FINITE ELEMENT METHOD

DATE : ? 2003 TIME : ?

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $\Omega_1 \equiv (-1,0)$, $\Omega_2 \equiv (0,1)$ and $\Omega \equiv (-1,1)$. Given $f \in L^2(\Omega)$ and strictly positive constants σ_1, σ_2 and c; consider the interface problem:

(P1) Find $u(x) := \begin{cases} u_1(x) & x \in \overline{\Omega}_1 \equiv [-1,0] \\ u_2(x) & x \in \overline{\Omega}_2 \equiv [0,1] \end{cases}$ such that

$$-\sigma_i\, u_i'' + c\, u_i = f \qquad \text{in } \Omega_i, \quad i = 1 \to 2;$$

$$u_1(-1) = u_2(1) = 0, \qquad u_1(0) = u_2(0) \qquad \text{and} \qquad \sigma_1\, u_1'(0) = \sigma_2\, u_2'(0).$$

Show that a solution of (P1) solves

(P2) Find $u \in H_0^1(\Omega) := \{\, w \in H^1(\Omega) \ : \ w(-1) = w(1) = 0 \,\}$ such that

$$a(u,v) = \ell(v) \qquad \forall\, v \in H_0^1(\Omega);$$

where for all v, $w \in H_0^1(\Omega)$

$$\ell(v) := \int_\Omega f\, v \,\mathrm{d}x, \qquad a(w,v) := \int_\Omega [\, \sigma\, w'\, v' + c\, w\, v \,] \,\mathrm{d}x$$

$$\text{and} \qquad \sigma(x) := \begin{cases} \sigma_1 & x \in \Omega_1 \\ \sigma_2 & x \in \Omega_2 \end{cases}.$$

Show that problem (P2) is *equivalent* to

(P3) Find $u \in H_0^1(\Omega)$ such that

$$a(u,u) - 2\,\ell(u) \leq a(v,v) - 2\,\ell(v) \qquad \forall\, v \in H_0^1(\Omega).$$

Show that the solution of (P2), and hence (P1), is unique.

2. If $w \in H^2(0,1)$ and $w(0) = w(1) = 0$, show that

$$\int_0^1 w^2 \,\mathrm{d}x \le \tfrac{1}{8} \int_0^1 (w')^2 \,\mathrm{d}x \qquad \text{and} \qquad \int_0^1 (w')^2 \,\mathrm{d}x \le \tfrac{1}{2} \int_0^1 (w'')^2 \,\mathrm{d}x.$$

Let

$$S^h := \left\{ v^h \in C[a,b] : v^h \text{ linear on } [x_{j-1}, x_j],\ j = 1 \to J \right\},$$

where $a = x_0 < x_1 \cdots < x_{j-1} < x_j \cdots < x_{J-1} < x_J = b$. For any $v \in H^2(a,b)$, let $v_I^h \in S^h$ be such that $v_I^h(x_j) = v(x_j)$, $j = 0 \to J$. Use the results above to show that for all $v \in H^2(a,b)$ and for $j = 1 \to J$ that

$$\int_{x_{j-1}}^{x_j} [(v - v_I^h)']^2 \,\mathrm{d}x \le \tfrac{1}{2}\, h_j^2 \int_{x_{j-1}}^{x_j} (v'')^2 \,\mathrm{d}x$$

$$\text{and} \qquad \int_{x_{j-1}}^{x_j} (v - v_I^h)^2 \,\mathrm{d}x \le \tfrac{1}{16}\, h_j^4 \int_{x_{j-1}}^{x_j} (v'')^2 \,\mathrm{d}x;$$

where $h_j := x_j - x_{j-1}$.

For all $f \in L^2(a,b)$, assume there exists a solution $u \in H^2(a,b)$, dependent on f, to the problem

$$-u'' + u = f \qquad \text{in } (a,b), \qquad\qquad u'(a) = u'(b) = 0.$$

By considering its weak formulation show that for any given f, the solution u is unique and that

$$\int_a^b [\,(u')^2 + u^2\,] \,\mathrm{d}x \le \int_a^b f^2 \,\mathrm{d}x.$$

Deduce from the differential equation and the above that

$$\int_a^b (u'')^2 \,\mathrm{d}x \le 4 \int_a^b f^2 \,\mathrm{d}x.$$

Formulate the finite element approximation $u^h \in S^h$ to the above problem. Show that for any given $f \in L^2(a,b)$, u^h exists and is unique.

Setting $e := u - u^h$, show that there exists a positive constant C such that

$$\int_a^b [\,(e')^2 + e^2\,] \,\mathrm{d}x \le C\, h^2 \int_a^b f^2 \,\mathrm{d}x,$$

where $h := \max_{j=1\to J} h_j$.

3. Let τ be a triangle with vertices P_1, P_2 and P_3. For $i = 1 \to 3$, let $\phi_i(x, y)$ be the linear function such that

$$\phi_i(P_j) \;=\; \delta_{i,j} \quad j = 1 \to 3.$$

State, without proof, the entries

$$\int_\tau \underline{\nabla}\phi_i \,.\, \underline{\nabla}\phi_j \;\mathrm{d}x\,\mathrm{d}y \qquad\qquad i, j = 1 \to 3$$

of the "element stiffness matrix" for τ in terms of the cotangents of its angles.

Consider the problem: Find u such that

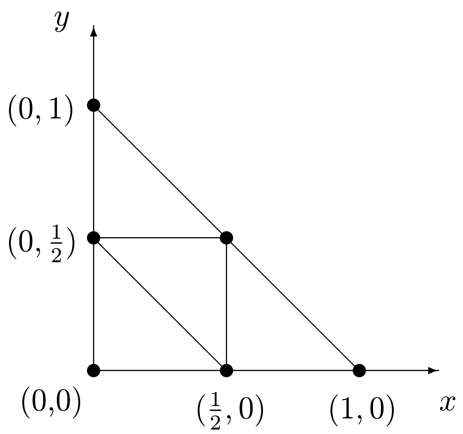
$$\nabla^2 u \;=\; 4 \qquad \text{in the triangle} \quad 0 < y < 1 - x, \quad 0 < x < 1;$$

subject to the boundary conditions

$$u(x, 1-x) = 2\,x^2, \qquad \frac{\partial u}{\partial y}(x, 0) = -2 \qquad \text{for} \quad 0 \le x \le 1;$$

$$\text{and} \qquad\qquad \frac{\partial u}{\partial x}(0, y) = 0 \qquad \text{for} \quad 0 \le y \le 1.$$

Formulate and compute the continuous piecewise linear approximation to the above problem based on the triangulation given in the figure below, where the nodes $1 \to 6$ have (x, y) coordinates $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$ respectively.



4. Let $\widehat{e}$ be the tetrahedron in $(\widehat{x}, \widehat{y}, \widehat{z})$ space with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. Consider the quadrature rule

$$\widehat{Q}_{\widehat{e}}(\widehat{v}) := \tfrac{1}{24}\left[\,\widehat{v}(\beta,\beta,\beta) + \widehat{v}(\alpha,\beta,\beta) + \widehat{v}(\beta,\alpha,\beta) + \widehat{v}(\beta,\beta,\alpha)\,\right]$$

approximating

$$\int_{\widehat{e}} \widehat{v}(\widehat{x},\widehat{y},\widehat{z})\, \mathrm{d}\widehat{x}\, \mathrm{d}\widehat{y}\, \mathrm{d}\widehat{z}.$$

Find bounds on α and β so that the sampling points of $\widehat{Q}_{\widehat{e}}(\cdot)$ are in $\widehat{e}$.

Find necessary and sufficient conditions for $\widehat{Q}_{\widehat{e}}(\widehat{v})$ to be exact

(i) for all $\widehat{v} \in \mathcal{P}_1(\widehat{x},\widehat{y},\widehat{z})$,

(ii) for all $\widehat{v} \in \mathcal{P}_2(\widehat{x},\widehat{y},\widehat{z})$;

where

$$\mathcal{P}_k(\widehat{x},\widehat{y},\widehat{z}) := \{\,\text{all polynomials in } \widehat{x},\widehat{y} \text{ and } \widehat{z} \text{ of degree } \le k\,\}.$$

[*You may use the result that*

$$\int_{\widehat{e}} \widehat{x}^i\, \widehat{y}^j\, \widehat{z}^k\, \mathrm{d}\widehat{x}\, \mathrm{d}\widehat{y}\, \mathrm{d}\widehat{z} = \frac{i!\; j!\; k!}{(i+j+k+3)!} \quad \forall\; i,\; j,\; k \in \mathbb{N}.$$]

Show that (ii) leads to a unique choice of α and β if the sampling points of $\widehat{Q}_{\widehat{e}}(\cdot)$ are required to be in $\widehat{e}$.

Find explicitly this choice.

Let e be the tetrahedron with vertices P_i, having coordinates (x_i, y_i, z_i), $i = 1 \to 4$. Derive a quadrature rule approximating

$$\int_e v(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

which is exact for all $v \in \mathcal{P}_2(x,y,z)$. State precisely the sampling points in terms of the coordinates (x_i, y_i, z_i), $i = 1 \to 4$, and the weights in terms of the volume of e.

5. Let $\widehat{e}$ be the square in the $(\widehat{x}, \widehat{y})$ plane with vertices $(-1,-1)$, $(1,-1)$, $(-1,1)$ and $(1,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. For $i = 1 \to 4$, find the bilinear functions $\widehat{\phi}_i(\widehat{x}, \widehat{y})$, $(\widehat{x}, \widehat{y}) \in \widehat{e}$, such that

$$\widehat{\phi}_i(\widehat{P}_j) \;=\; \delta_{ij} \qquad j = 1 \to 4.$$

Given points P_i with coordinates (x_i, y_i), $i = 1 \to 4$, such that $x_1 = y_1 = y_2 = x_3 = 0$ and $x_2,\ x_4,\ y_3,\ y_4 > 0$; consider the mapping $F : (\widehat{x}, \widehat{y}) \in \widehat{e} \to (x, y)$ defined by

$$x \;=\; \sum_{i=1}^{4} x_i\, \widehat{\phi}_i(\widehat{x}, \widehat{y}) \quad \text{and} \quad y \;=\; \sum_{i=1}^{4} y_i\, \widehat{\phi}_i(\widehat{x}, \widehat{y}).$$

Sketch the image, e, of $\widehat{e}$ under the map F.

Show that $F : \widehat{e} \to e$ is invertible if

$$x_2\, y_4 + x_4\, y_3 > x_2\, y_3.$$

Show that this condition is equivalent to the point P_4 being above the straight line joining P_2 to P_3; that is, e is a convex quadrilateral.

Find the area of e.

Assuming that F is invertible; find $\phi_i(\frac{1}{4}(x_2 + x_4), \frac{1}{4}(y_3 + y_4))$, where

$$\phi_i(x, y) := \widehat{\phi}_i(F^{-1}(x, y)) \quad \forall\, (x, y) \in e, \quad i = 1 \to 4.$$