UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION(MATHEMATICS) MAY-JUNE 2002

This paper is also taken for the relevant examination for the Associateship

M3N8/M4N8/MSA8 FINITE ELEMENT METHOD

DATE : Tuesday 28 May 2002 TIME : 2.00pm – 4.00pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Given α, β, g_i, $i = 1 \to 4$, $\in \mathbb{R}$, $\sigma \in C^2[0,1]$, $b \in C^1[0,1]$ and c, $f \in C[0,1]$; where

$$\sigma(x) \geq \sigma_0 > 0 \quad \text{and} \quad b(x),\, c(x),\, \alpha,\, \beta \geq 0 \quad \forall\, x \in [0,1];$$

consider the following problem:

(P1) Find u such that

$$(\sigma\, u'')'' - (b\, u')' + c\, u \;=\; f \quad \text{in } (0,1),$$

$$u(0) = g_1, \quad u'(1) = g_2, \quad [\sigma\, u'' - \alpha\, u'](0) = g_3$$

$$\text{and} \quad [(\sigma\, u'')' - b\, u' - \beta\, u](1) = g_4.$$

Let

$$V(g_1, g_2) := \left\{ v \in H^2(0,1) \;:\; v(0) = g_1 \quad \text{and} \quad v'(1) = g_2 \right\}.$$

Show that a solution of (P1) is a solution of the following problem:

(P2) Find $u \in V(g_1, g_2)$ such that

$$a(u, v) \;=\; \ell(v) \qquad \forall\, v \in V(0,0);$$

where for all w, $v \in H^2(0,1)$

$$a(w, v) := \int_0^1 [\sigma\, w''\, v'' + b\, w'\, v' + c\, w\, v]\, \mathrm{d}x \;+\; \alpha\, w'(0)\, v'(0) + \;\beta\, w(1)\, v(1),$$

$$\ell(v) := \int_0^1 f\, v\, \mathrm{d}x \;-\; g_3\, v'(0) - \; g_4\, v(1).$$

Show that problem (P2) is *equivalent* to the following problem:

(P3) Find $u \in V(g_1, g_2)$ such that

$$a(u, u) \;- 2\,\ell(u) \;\leq\; a(w, w) \;- 2\,\ell(w) \qquad \forall\, w \in V(g_1, g_2).$$

Show that the solution of (P2), and hence (P1), is unique.

2. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal region with boundary $\partial\Omega$. For $m \in \mathbb{N}$, let $H^m(\Omega)$ be the Sobolev space with norm

$$\|v\|_{m,\Omega} := \left\{ \sum_{i,j\,:\,0 \le i+j \le m} \int_\Omega \left[\frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right]^2 \mathrm{d}x\,\mathrm{d}y \right\}^{1/2}.$$

Given $\sigma \in C^1(\overline{\Omega})$ and $c \in C(\overline{\Omega})$, where

$$\sigma(x,y) \ge \sigma_0 > 0 \quad \text{and} \quad c(x,y) \ge c_0 > 0 \quad \forall\,(x,y) \in \overline{\Omega},$$

let A be the differential operator

$$A\,v := -\underline{\nabla}\,.\,(\sigma\,\underline{\nabla} v) \;+\; c\,v.$$

For all $f \in L^2(\Omega)$ assume there exists a solution $w \in H^2(\Omega)$, dependent on f, to the problem

$$A\,w = f \quad \text{in } \Omega, \qquad\qquad w = 0 \quad \text{on } \partial\Omega;$$

and that

$$\|w\|_{2,\Omega} \le M_1\,[\,\|f\|_{0,\Omega} \;+\; \|w\|_{1,\Omega}],$$

where M_1 is a positive constant depending only on σ, c and Ω. By considering its weak formulation show that w is unique and that

$$\|w\|_{2,\Omega} \le M_2\,\|f\|_{0,\Omega},$$

where M_2 is a positive constant depending only on σ, c and Ω.

Let T^h be a partitioning of Ω into regular triangles τ with maximum diameter h. Explain briefly what is meant by the term regular.

Define

$$S^h := \{\, v^h \in C(\overline{\Omega}) : v^h \text{ linear on } \tau,\ \forall\,\tau \in T^h\}.$$

Formulate the finite element approximation $w^h \in S^h$ to the above problem. Show that for any given $f \in L^2(\Omega)$, w^h exists and is unique.

Defining $e := w - w^h$, show that there exists a positive constant C_1 such that

$$\|e\|_{1,\Omega} \le C_1\,h\,\|f\|_{0,\Omega}.$$

[You may use the result that for all $v \in H^2(\Omega)$ there exists $v_I^h \in S^h$ such that

$$\|v - v_I^h\|_{1,\Omega} \le C\,h\,\|v\|_{2,\Omega},$$

where C is a positive constant independent of v and h].

By considering the weak formulation of the auxiliary problem: find z such that

$$A\,z = e \quad \text{in } \Omega, \qquad\qquad z = 0 \quad \text{on } \partial\Omega;$$

show that there exists a positive constant C_2 such that

$$\|e\|_{0,\Omega} \le C_2\,h^2\,\|f\|_{0,\Omega}.$$

3. Let τ be a triangle with vertices P_1, P_2 and P_3. For $i = 1 \to 3$, let $\phi_i(x,y)$ be the linear function such that

$$\phi_i(P_j) \;=\; \delta_{i,j} \quad j = 1 \to 3.$$

State, without proof, the entries

$$\int_\tau \underline{\nabla}\phi_i \,.\, \underline{\nabla}\phi_j \;\mathrm{d}x\,\mathrm{d}y \qquad\qquad i,j = 1 \to 3$$

of the "element stiffness matrix" for τ in terms of the cotangents of its angles.

Consider the problem: Find u such that

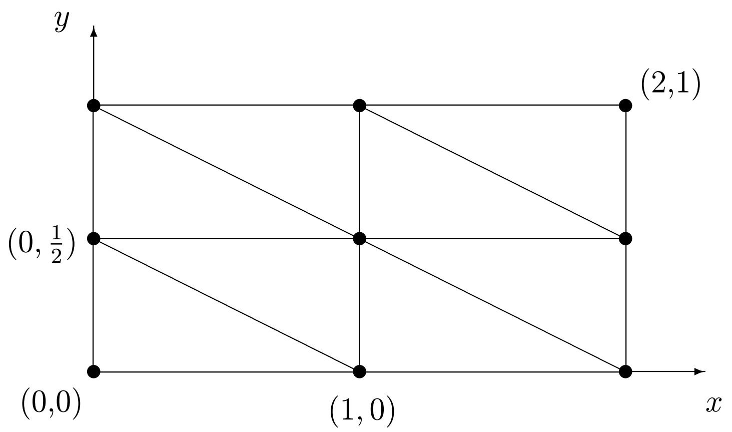
$$-\nabla^2 u \;=\; 2 \qquad \text{in the rectangle} \quad 0 < x < 2, \quad 0 < y < 1;$$

subject to the boundary conditions

$$u(x,0) = x^2 + x, \qquad u(x,1) = x^2 + x - 2 \qquad \text{for} \quad 0 \le x \le 2;$$

$$\frac{\partial u}{\partial x}(0,y) = 1, \qquad u(2,y) = 6 - 2\,y^2 \qquad \text{for} \quad 0 \le y \le 1.$$

Formulate and compute the continuous piecewise linear approximation to the above problem based on the triangulation given in the figure below, where the nodes $1 \to 9$ have (x,y) coordinates $(0,0)$, $(1,0)$, $(2,0)$, $(0,\frac{1}{2})$, $(1,\frac{1}{2})$, $(2,\frac{1}{2})$, $(0,1)$, $(1,1)$ and $(2,1)$ respectively.



4. Let $\widehat{e}$ be the tetrahedron in $(\widehat{x},\widehat{y},\widehat{z})$ space with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively.

Let

$$\widehat{P}_{123} := \tfrac{1}{3}\,[\widehat{P}_1+\widehat{P}_2+\widehat{P}_3], \qquad \widehat{P}_{124} := \tfrac{1}{3}\,[\widehat{P}_1+\widehat{P}_2+\widehat{P}_4],$$
$$\widehat{P}_{134} := \tfrac{1}{3}\,[\widehat{P}_1+\widehat{P}_3+\widehat{P}_4], \qquad \widehat{P}_{234} := \tfrac{1}{3}\,[\widehat{P}_2+\widehat{P}_3+\widehat{P}_4].$$

Consider the following quadrature rules

$$\widehat{Q}^{(1)}_{\widehat{e}}(\widehat{v}) := \tfrac{1}{24}\left[\widehat{v}(\widehat{P}_1)+\widehat{v}(\widehat{P}_2)+\widehat{v}(\widehat{P}_3)+\widehat{v}(\widehat{P}_4)\right],$$
$$\widehat{Q}^{(2)}_{\widehat{e}}(\widehat{v}) := \tfrac{1}{24}\left[\widehat{v}(\widehat{P}_{123})+\widehat{v}(\widehat{P}_{124})+\widehat{v}(\widehat{P}_{134})+\widehat{v}(\widehat{P}_{234})\right],$$

approximating

$$\int_{\widehat{e}} \widehat{v}(\widehat{x},\widehat{y},\widehat{z})\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}\,\mathrm{d}\widehat{z}.$$

Show that $\widehat{Q}^{(1)}_{\widehat{e}}(\widehat{v})$ and $\widehat{Q}^{(2)}_{\widehat{e}}(\widehat{v})$ are exact for all $\widehat{v}\in\mathcal{P}_1(\widehat{x},\widehat{y},\widehat{z})$, where

$$\mathcal{P}_k(\widehat{x},\widehat{y},\widehat{z}) := \{\text{ all polynomials in } \widehat{x},\widehat{y} \text{ and } \widehat{z} \text{ of degree } \le k\,\}.$$

[*You may use the result that*

$$\int_{\widehat{e}} \widehat{x}^i\widehat{y}^j\widehat{z}^k\,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}\,\mathrm{d}\widehat{z} = \frac{i!\;j!\;k!}{(i+j+k+3)!} \qquad \forall\; i,\,j,\,k\in\mathbb{N}.$$]

Find ω such that

$$\omega\,\widehat{Q}^{(1)}_{\widehat{e}}(\widehat{v}) + (1-\omega)\,\widehat{Q}^{(2)}_{\widehat{e}}(\widehat{v})$$

is exact for all $\widehat{v}\in\mathcal{P}_2(\widehat{x},\widehat{y},\widehat{z})$.

Show that this quadrature rule is also exact for all $\widehat{v}\in\mathcal{P}_3(\widehat{x},\widehat{y},\widehat{z})$.

Let e be the tetrahedron with vertices P_i, having coordinates (x_i,y_i,z_i), $i = 1\to 4$. Derive a quadrature rule approximating

$$\int_e v(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

which is exact for all $v\in\mathcal{P}_3(x,y,z)$. State precisely the sampling points in terms of the coordinates (x_i,y_i,z_i), $i=1\to 4$, and the weights in terms of the volume of e.

5. Let $\widehat{e}$ be the square in the $(\widehat{x}, \widehat{y})$ plane with vertices $(-1,-1)$, $(1,-1)$, $(-1,1)$ and $(1,1)$ labelled $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$ and $\widehat{P}_4$ respectively. In addition there are nodes $\widehat{P}_5$, $\widehat{P}_6$, $\widehat{P}_7$ and $\widehat{P}_8$ on $\widehat{e}$ with coordinates $(0,-1)$, $(-1,0)$, $(1,0)$ and $(0,1)$ respectively. Let B be the set of functions defined on $\widehat{e}$ such that

$$f \in B \Longrightarrow f(\widehat{x}, \widehat{y}) = a_1 + a_2\,\widehat{x} + a_3\,\widehat{y} + a_4\,\widehat{x}^2 + a_5\,\widehat{x}\,\widehat{y} + a_6\,\widehat{y}^2 + a_7\,\widehat{x}^2\,\widehat{y} + a_8\,\widehat{x}\,\widehat{y}^2$$

for some constants $\{a_i\}_{i=1}^8$. Let $\{\widehat{\phi}_i\}_{i=1}^8$ be the basis functions such that

$$\widehat{\phi}_i \in B \qquad \text{and} \qquad \widehat{\phi}_i(\widehat{P}_j) \;=\; \delta_{i,j} \qquad i,\, j = 1 \rightarrow 8.$$

Find $\widehat{\phi}_8$.

Let the points P_j have coordinates (x_j, y_j), $j = 1 \rightarrow 8$, such that $P_j \equiv \widehat{P}_j$ for $j = 1 \rightarrow 7$; $|x_8| < \frac{1}{2}$ and $y_8 > 0$. Consider the mapping $F\,:(\widehat{x}, \widehat{y}) \in \widehat{e} \rightarrow (x, y)$ given by

$$x \;=\; \sum_{i=1}^{8} x_i\,\widehat{\phi}_i(\widehat{x}, \widehat{y}) \quad \text{and} \quad y \;=\; \sum_{i=1}^{8} y_i\,\widehat{\phi}_i(\widehat{x}, \widehat{y}).$$

Sketch the image, e, of $\widehat{e}$ under the map F.

Show that F is invertible.

Find $\phi_8(\frac{1}{2}\,x_8,\, \frac{1}{2}\,(y_8 - 1))$ and $\underline{\nabla}\phi_8(\frac{1}{2}\,x_8,\, \frac{1}{2}\,(y_8 - 1))$, where

$$\phi_8(x, y) := \widehat{\phi}_8(F^{-1}(x, y)) \qquad \forall\; (x, y) \in e.$$